ECED 4601 Digital Control Systems

Midterm Reference Solution

Date: October 27 2017 **Room: MA 120 & B311**

Student name	
ID	

1) Obtain the inverse z transform of $X(z)=\frac{0.368z^2+0.478z+0.154}{(z-1)z^2}$

Using inversion integral method

Solution:

$$X(z) = \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z^2}$$

$$X(z)z^{k-1} = \frac{(0.368z^2 + 0.478z + 0.154)z^{k-1}}{(z-1)z^2}$$

For k = 0:

$$X(z)z^{k-1} = \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z^3}$$

$$x(0) = \left[\text{residue of } \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z^3} \text{ at triple pole } z = 0\right]$$

$$+ \left[\text{residue of } \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z^3} \text{ at pole } z = 1\right]$$

$$= \frac{1}{(3-1)!} \lim_{z \to 0} \frac{d^2}{dz^2}\left[\frac{0.368z^2 + 0.478z + 0.154}{z-1}\right]$$

$$+ \lim_{z \to 1}\left[\frac{0.368z^2 + 0.478z + 0.154}{z^3}\right] = -1 + 1 = 0$$

For k = 1:

$$X(z)z^{k-1} = \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z^2}$$

$$x(1) = \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d}{dz}\left[\frac{0.368z^2 + 0.478z + 0.154}{z-1}\right]$$

$$+ \lim_{z \to 1}\left[\frac{0.368z^2 + 0.478z + 0.154}{z^2}\right] = -0.632 + 1 = 0.368$$

For k = 2:

$$X(z)z^{k-1} = \frac{0.368z^2 + 0.478z + 0.154}{(z-1)z}$$

$$x(2) = \lim_{z \to 0} \left[\frac{0.368z^2 + 0.478z + 0.154}{z - 1} \right]$$

$$+ \lim_{z \to 1} \left[\frac{0.368z^2 + 0.478z + 0.154}{z} \right] = -0.154 + 1 = 0.846$$

For k = 3, 4, 5, ...:

$$X(z)z^{k-1} = \frac{(0.368z^2 + 0.478z + 0.154)z^{k-3}}{z - 1}$$

$$x(k) = \lim_{z \to 1} \left[(0.368z^2 + 0.478z + 0.154)z^{k-3} \right] = 1$$

In summary, we have

$$x(0) = 0$$

$$x(1) = 0.368$$

$$x(2) = 0.846$$

$$x(k) = 1 \qquad \text{for } k = 3, 4, 5, \ldots$$

2) Consider the difference equation in a closed form.:

$$x(k+2) - 1.3679x(k+1) + 0.3679x(k) = 0.3679u(k+1) + 0.2642u(k)$$

Where $x(k) = 0 \;\; for\; k \le 0$

The input function $u(k)$ is given by:

$$u(k) = 0, \qquad k < 0$$

$$u(0) = 1.5820$$

$$u(1) = -0.5820$$

$$u(k) = 0, \qquad k = 2, 3, 4, \cdots$$

Determine the output $x(k)$

Solution:

$$x(k+2) - 1.3679x(k+1) + 0.3679x(k) = 0.3679u(k+1) + 0.2642u(k)$$

The z transform of this equation is

$$z^2X(z) - z^2x(0) - zx(1) - 1.3679\left[zX(z) - zx(0)\right] + 0.3679X(z)$$
$$= 0.3679\left[zU(z) - zu(0)\right] + 0.2642\,U(z)$$

Noting that $x(0) = 0$ and $x(1) = 0.5820$, we have

$$(z^2 - 1.3679z + 0.3679)X(z) = (0.3679z + 0.2642)U(z)$$

or

$$\frac{X(z)}{U(z)} = \frac{0.3679z^{-1} + 0.2642z^{-2}}{1 - 1.3679z^{-1} + 0.3679z^{-2}}$$

Since

$$U(z) = 1.5820 - 0.5820z^{-1}$$

we have

$$X(z) = \frac{0.5820z^{-1} + 0.2038z^{-2} - 0.1538z^{-3}}{1 - 1.3679z^{-1} + 0.3679z^{-2}}$$

$$= 0.5820z^{-1} + z^{-2} + z^{-3} + \cdots$$

Hence

$$x(0) = 0$$
$$x(1) = 0.5820$$
$$x(k) = 1 \qquad \text{for } k = 2, 3, 4, \ldots$$

3) Obtain the inverse z transform of $X(s) = \dfrac{k}{(s+a)(s+b)}$

a) Using the residue method
b) Using the method based on impulse response function

Solution:

1. <u>Residue method:</u>

$$X(z) = \left[\text{residue of } \frac{X(s)z}{z - e^{Ts}} \text{ at pole } s = -a\right] + \left[\text{residue of } \frac{X(s)z}{z - e^{Ts}} \text{ at pole } s = -b\right]$$

$$= \lim_{s \to -a}\left[(s + a)\frac{K}{(s + a)(s + b)}\frac{z}{z - e^{Ts}}\right] + \lim_{s \to -b}\left[(s + b)\frac{K}{(s + a)(s + b)}\frac{z}{z - e^{Ts}}\right]$$

$$= \frac{K}{b - a}\frac{z}{z - e^{-aT}} + \frac{K}{a - b}\frac{z}{z - e^{-bT}}$$

$$= \frac{K}{b - a}\left(\frac{1}{1 - e^{-aT}z^{-1}} - \frac{1}{1 - e^{-bT}z^{-1}}\right)$$

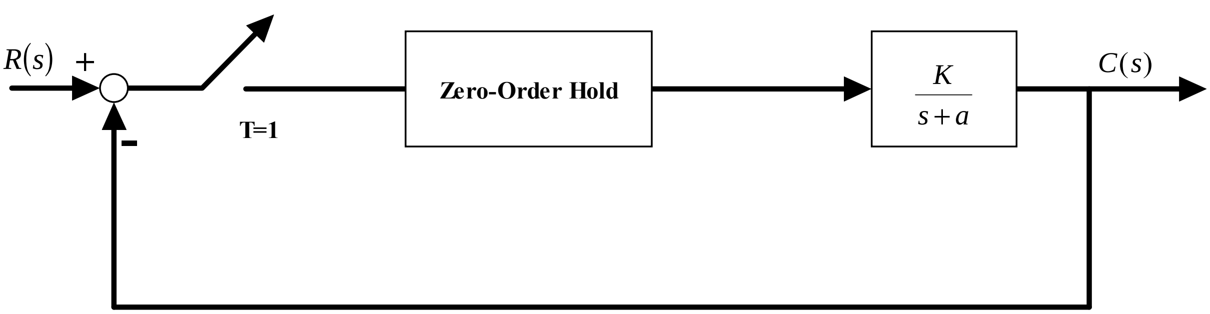
2. <u>Method based on impulse response function:</u>

$$x(t) = \frac{K}{b - a}(e^{-at} - e^{-bt})$$

Hence

$$X(z) = \frac{K}{b - a}\left(\frac{1}{1 - e^{-aT}z^{-1}} - \frac{1}{1 - e^{-bT}z^{-1}}\right)$$

4) Obtain the close loop pulse transfer function of the system shown below:



$$G(z) = \mathcal{Z}\left[G_{h0}(s)\frac{K}{s + a}\right] = (1 - z^{-1})\,\mathcal{Z}\left[\frac{K}{s(s + a)}\right]$$

$$= \frac{K}{a}\frac{(1 - e^{-aT})z^{-1}}{1 - e^{-aT}z^{-1}} = \frac{K}{a}\frac{(1 - e^{-a})z^{-1}}{1 - e^{-a}z^{-1}} \qquad (T = 1)$$

Then

$$\frac{C(z)}{R(z)} = \frac{G(z)}{1 + G(z)} = \frac{K(1 - e^{-a})z^{-1}}{a + \left[K - (K + a)e^{-a}\right]z^{-1}}$$

5) Determine the stability of the following discrete-time system using Jury stability criterion.

$$\frac{Y(z)}{X(z)} = \frac{z^{-3}}{1 + 0.5z^{-1} - 1.34z^{-2} + 0.24z^{-3}}$$

Solution:

$$\frac{Y(z)}{X(z)} = \frac{1}{z^3 + 0.5z^2 - 1.34z + 0.24}$$

Define

$$P(z) = z^3 + 0.5z^2 - 1.34z + 0.24$$
$$= a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

Then

$$a_0 = 1$$
$$a_1 = 0.5$$
$$a_2 = -1.34$$
$$a_3 = 0.24$$

The Jury stability conditions are

1. $|a_3| < a_0$

   This condition is satisfied.

2. $P(1) > 0$

   Since

   $$P(1) = 1 + 0.5 - 1.34 + 0.24 = 0.4 > 0$$

   the condition is satisfied.

3. $P(-1) < 0$

   Since

   $$P(-1) = -1 + 0.5 + 1.34 + 0.24 = 1.08 > 0$$

   the condition is not satisfied.

4. $|b_2| > |b_0|$

   Since condition (3) is not satisfied (the system is unstable), it is not necessary to test condition (4).

The conclusion is that the system is unstable.

Laplace Transform Pairs for Bilateral

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\mathrm{Re}(s) < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\mathrm{Re}(s) > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\mathrm{Re}(s) < 0$
6	$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > -a$
7	$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < -a$
8	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$\mathrm{Re}(s) > -a$
9	$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$\mathrm{Re}(s) < -a$
10	$\delta(t-T)$	e^{-sT}	All s
11	$[\cos w_0 t]u(t)$	$\frac{s}{s^2+w_0^2}$	$\mathrm{Re}(s) > 0$
12	$[\sin w_0 t]u(t)$	$\frac{w_0}{s^2+w_0^2}$	$\mathrm{Re}(s) > 0$
13	$[e^{-at}\cos w_0 t]u(t)$	$\frac{s+a}{(s+a)^2+w_0^2}$	$\mathrm{Re}(s) > -a$
14	$[e^{-at}\sin w_0 t]u(t)$	$\frac{w_0}{(s+a)^2+w_0^2}$	$\mathrm{Re}(s) > -a$
15	$u_n(t) = \frac{d^n\delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = u(t) * \cdots * u(t)$	$\frac{1}{s^n}$	$\mathrm{Re}(s) > 0$

Laplace Transform Property for Bilateral

Property	Signal	Laplace Transform	ROC
	$x(t)$, $x_1(t)$, $x_2(t)$	$X(s)$, $X_1(s)$, $X_2(s)$	R, R_1, R_2
Linearity	$ax_1(t)+bx_2(t)$	$aX_1(S)+bX_2(S)$	$R_1 \cap R_2$
Time Shifting	$x(t-t_0)$	$e^{-st_0}X(S)$	R
Shift in the s-Domain	$e^{s_0 t}x(t)$	$X(s-s_0)$	R
Time Scaling	$x(at)$	$\frac{1}{\lvert a \rvert}X(\frac{s}{a})$	Scaled ROC
Conjugation	$x^*(t)$	$X^*(S^*)$	R
Convolution	$x_1(t)*x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
Differentiation in the time-Domain	$\frac{dx(t)}{dt}$	$SX(S)$	At least R
Differentiation in the s-Domain	$-tx(t)$	$\frac{dX(S)}{dS}$	R
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{S}X(S)$	At least $R \cap \{\text{Re}(s)>0\}$
Initial value theorem	$x(0^+) = \lim_{s\to\infty} SX(S)$		
Final value theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} SX(S)$		

Laplace Transform Property for Unilateral

Property	Signal	Laplace Transform
	$x(t)$, $x_1(t)$, $x_2(t)$	$\chi(s)$, $\chi_1(s)$, $\chi_2(s)$
Linearity	$ax_1(t)+bx_2(t)$	$a\chi_1(S)+b\chi_2(S)$
Shift in the s-Domain	$e^{s_0 t}x(t)$	$\chi(s-s_0)$
Time Scaling	$x(at)$	$\frac{1}{\lvert a \rvert}\chi(\frac{s}{a})$
Conjugation	$x^*(t)$	$\chi^*(S)$
Convolution	$x_1(t)*x_2(t)$	$\chi_1(s)\chi_2(s)$
Differentiation in the time-Domain	$\frac{dx(t)}{dt}$	$S\chi(S)-x(0^-)$
	$\frac{d^2x(t)}{dt^2}$	$S^2\chi(S)-sx(0^-)-x'(0^-)$
Differentiation in the s-Domain	$-tx(t)$	$\frac{d\chi(S)}{dS}$
Integration in the Time Domain	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{S}\chi(S)$
Initial value theorem	$x(0^+) = \lim_{s\to\infty} S\chi(S)$	
Final value theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} S\chi(S)$	

Table of z transforms 1

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	—	—	Kronecker delta $\delta_0(k)$ $1, \quad k = 0$ $0, \quad k \neq 0$	1
2.	—	—	$\delta_0(n-k)$ $1, \quad n = k$ $0, \quad n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1-z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1-e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1-z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2 z^{-1}(1+z^{-1})}{(1-z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3 z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1-e^{-at}$	$1-e^{-akT}$	$\frac{(1-e^{-aT})z^{-1}}{(1-z^{-1})(1-e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	$e^{-akT}-e^{-bkT}$	$\frac{(e^{-aT}-e^{-bT})z^{-1}}{(1-e^{-aT}z^{-1})(1-e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	$(1-akT)e^{-akT}$	$\frac{1-(1+aT)e^{-aT}z^{-1}}{(1-e^{-aT}z^{-1})^2}$

Table of z transforms 2

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
12.	$\frac{2}{(s+a)^3}$	$t^2 e^{-at}$	$(kT)^2 e^{-akT}$	$\frac{T^2 e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at - 1 + e^{-at}$	$akT - 1 + e^{-akT}$	$\frac{[(aT-1+e^{-aT}) + (1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T + z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T + z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T + e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T + e^{-2aT}z^{-2}}$
18.			a^k	$\frac{1}{1-az^{-1}}$
19.			a^{k-1} $k = 1, 2, 3, \ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.			ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.			$k^2 a^{k-1}$	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.			$k^3 a^{k-1}$	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.			$k^4 a^{k-1}$	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.			$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$
25.			$\frac{k(k-1)}{2!}$	$\frac{z^{-2}}{(1-z^{-1})^3}$
26.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}$	$\frac{z^{-m+1}}{(1-z^{-1})^m}$
27.			$\frac{k(k-1)}{2!}a^{k-2}$	$\frac{z^{-2}}{(1-az^{-1})^3}$
28.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}a^{k-m+1}$	$\frac{z^{-m+1}}{(1-az^{-1})^m}$

Property table:

	$x(t)$ or $x(k)$	$\mathcal{Z}[x(t)]$ or $\mathcal{Z}[x(k)]$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t) + bx_2(t)$	$aX_1(z) + bX_2(z)$
3.	$x(t+T)$ or $x(k+1)$	$zX(z) - zx(0)$
4.	$x(t+2T)$	$z^2X(z) - z^2x(0) - zx(T)$
5.	$x(k+2)$	$z^2X(z) - z^2x(0) - zx(1)$
6.	$x(t+kT)$	$z^kX(z) - z^kx(0) - z^{k-1}x(T) - \cdots - zx(kT-T)$
7.	$x(t-kT)$	$z^{-k}X(z)$
8.	$x(n+k)$	$z^kX(z) - z^kx(0) - z^{k-1}x(1) - \cdots - zx(k-1)$
9.	$x(n-k)$	$z^{-k}X(z)$
10.	$tx(t)$	$-Tz\frac{d}{dz}X(z)$
11.	$kx(k)$	$-z\frac{d}{dz}X(z)$
12.	$e^{-at}x(t)$	$X(ze^{aT})$
13.	$e^{-ak}x(k)$	$X(ze^{a})$
14.	$a^kx(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^kx(k)$	$-z\frac{d}{dz}X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z\to\infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z\to 1} [(1-z^{-1})X(z)]$ if $(1-z^{-1})X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k) = x(k) - x(k-1)$	$(1-z^{-1})X(z)$
19.	$\Delta x(k) = x(k+1) - x(k)$	$(z-1)X(z) - zx(0)$
20.	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1-z^{-1}}X(z)$
21.	$\frac{\partial}{\partial a}x(t,a)$	$\frac{\partial}{\partial a}X(z,a)$
22.	$k^mx(k)$	$\left(-z\frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^{n} x(kT)y(nT-kT)$	$X(z)Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

Table: z transform of $x(k+m)$ and $x(k-m)$

Discrete function	z Transform
$x(k+4)$	$z^4X(z) - z^4x(0) - z^3x(1) - z^2x(2) - zx(3)$
$x(k+3)$	$z^3X(z) - z^3x(0) - z^2x(1) - zx(2)$
$x(k+2)$	$z^2X(z) - z^2x(0) - zx(1)$
$x(k+1)$	$zX(z) - zx(0)$
$x(k)$	$X(z)$
$x(k-1)$	$z^{-1}X(z)$
$x(k-2)$	$z^{-2}X(z)$
$x(k-3)$	$z^{-3}X(z)$
$x(k-4)$	$z^{-4}X(z)$