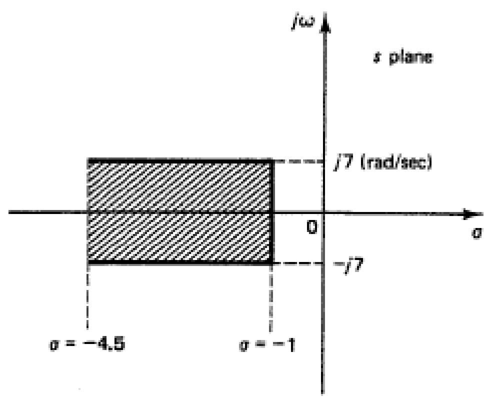
ECED 4601 Digital Control Systems

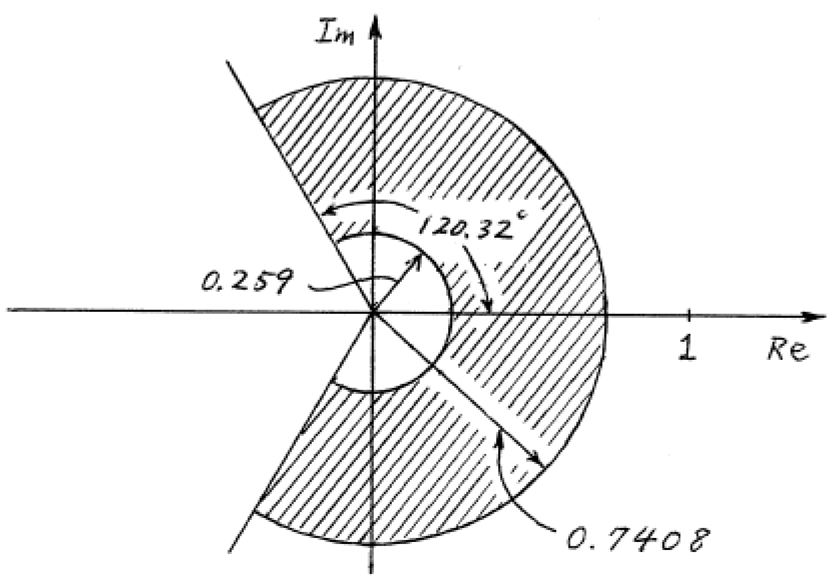
Assignment #3 reference solution

http://www.jasongu.org/4601/assignments.html

Assignment #3 contains the following problems:

1) Problem B-4-1: Consider the region in the s plane shown in figure. Draw the corresponding regions in the z plane. The sampling period T is assumed to be 0.3 second.





2) B-4-3 determine the stability of the following discrete-time system.

a) using Jury stability criterion

Define

$$P(z) = z^3 + 0.5z^2 - 1.34z + 0.24$$
$$= a_0 z^3 + a_1 z^2 + a_2 z + a_3$$

Then

$$a_0 = 1$$
$$a_1 = 0.5$$
$$a_2 = -1.34$$
$$a_3 = 0.24$$

The Jury stability conditions are

1. $|a_3| < a_0$

   This condition is satisfied.

2. $P(1) > 0$

   Since

   $$P(1) = 1 + 0.5 - 1.34 + 0.24 = 0.4 > 0$$

   the condition is satisfied.

3. $P(-1) < 0$

   Since

   $$P(-1) = -1 + 0.5 + 1.34 + 0.24 = 1.08 > 0$$

   the condition is not satisfied.

4. $|b_2| > |b_0|$

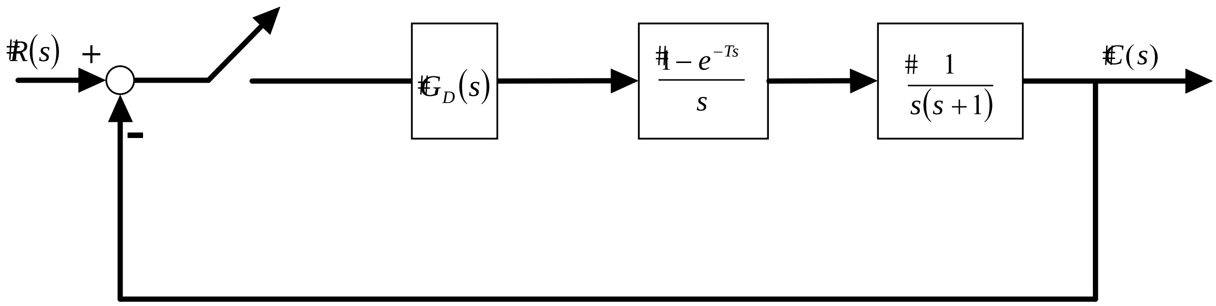
   Since condition (3) is not satisfied (the system is unstable), it is not necessary to test condition (4).

The conclusion is that the system is unstable.

b) Using the bilinear transformation coupled with the routh stabilioty criterion.

$$\frac{X(z)}{X(z)} = \frac{z^{-3}}{1 + 0.5z^{-1} - 1.34z^{-2} + 0.24z^{-3}}$$

3) B-4-9 Consider the system in figure, design a digital controller $G_D(z)$ such that the damping ratio ξ of the dominant closed loop pole sis 0.5 and the number of samples per cycle of damped sinusoidal oscillation is 8. assume that the sampling period is 0.1 second. Determine the static velocity error constant. Also, determine the response of the designed system to a unit step input.



$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s}\frac{1}{s(s+1)}\right] = (1 - z^{-1})\,\mathcal{Z}\left[\frac{1}{s^2(s+1)}\right]$$

$$= \frac{(T - 1 + e^{-T})z^{-1} + (1 - e^{-T} - Te^{-T})z^{-2}}{(1 - z^{-1})(1 - e^{-T}z^{-1})}$$

Since T = 0.1 sec, G(z) becomes

$$G(z) = \frac{0.004837z^{-1}(1 + 0.9674z^{-1})}{(1 - z^{-1})(1 - 0.9048z^{-1})} = \frac{0.004837(z + 0.9674)}{(z - 1)(z - 0.9048)}$$

Since the number of samples per cycle of damped sinusoidal oscillation is specified as 8, one of the dominant closed-loop poles must be on the line having an angle of 45° and passimg through the origin. Thus, the desired dominant closed-loop pole location in the upper half z plane can be determined as the intersection of the line having an angle of 45° and the $\zeta = 0.5$ locus. The equations for the constant ζ locus are

$$|z| = \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\frac{\omega_d}{\omega_s}\right)$$

and

$$\angle z = 2\pi\frac{\omega_d}{\omega_s}$$

Hence for $\zeta = 0.5$ and $\angle z = \frac{1}{4}\pi$, we have

$$\frac{\omega_d}{\omega_s} = \frac{1}{8}$$

and

$$|z| = \exp\left(-\frac{2\pi \times 0.5}{\sqrt{1-0.5^2}}\,\frac{1}{8}\right) = \exp\left(-\frac{\pi}{0.866}\,\frac{1}{8}\right) = 0.6354$$

The desired dominant closed-loop pole in the upper half z plane is located at

$$z = 0.6354\ \angle 45^\circ = 0.4493 + j0.4493$$

In order to have a closed-loop pole at this location, we need to add a phase lead angle of 78.59°. The digital controller must give this necessary phase lead angle.

We shall choose the digital controller $G_D(z)$ to be

$$G_D(z) = K\frac{z+\alpha}{z+\beta} = K\frac{z - 0.9048}{z + \beta}$$

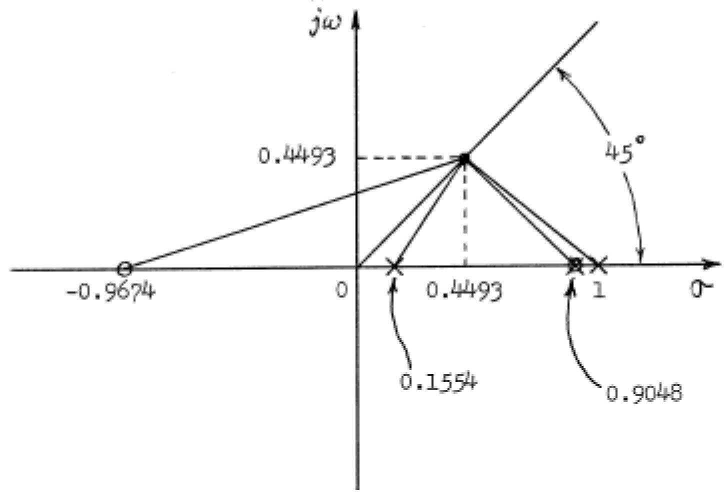
(Here we chose $\alpha = -0.9048$.) Then, from the angle condition we find that the controller pole must be located at $z = 0.1554$, or $\beta = -0.1554$. (See the diagram next page.)

The controller $G_D(z)$ is now given by

$$G_D(z) = K\frac{z - 0.9048}{z - 0.1554}$$

The open-loop pulse transfer function becomes

$$G_D(z)G(z) = K\frac{0.004837(z + 0.9674)}{(z - 0.1554)(z - 1)}$$



Using the magnitude condition, the gain K can be determined as follows:

$$\left| K \frac{0.004837(z + 0.9674)}{(z - 0.1554)(z - 1)} \right|_{z = 0.4493 + j0.4493} = 1$$

or

$$K = 53.08$$

Thus, the digital controller has the following pulse transfer function:

$$G_D(z) = 53.08 \frac{z - 0.9048}{z - 0.1554}$$

The static velocity error constant K_v is determined as follows:

$$K_v = \lim_{z \to 1} \frac{1 - z^{-1}}{0.1} (53.08) \frac{z - 0.9048}{z - 0.1554} \frac{(0.004837)(z + 0.9674)}{(z - 1)(z - 0.9048)}$$

$$= 5.98$$

We shall next obtain the unit-step response sequence. Since the open-loop pulse transfer function is

$$G_D(z)G(z) = 0.2567 \frac{z + 0.9674}{(z - 0.1554)(z - 1)}$$

the closed-loop pulse transfer function becomes

$$\frac{C(z)}{R(z)} = \frac{0.2567(z + 0.9674)}{(z - 0.1554)(z - 1) + 0.2567(z + 0.9674)}$$

$$= \frac{0.2567z^{-1} + 0.2483z^{-2}}{1 - 0.8987z^{-1} + 0.4037z^{-2}}$$

Since the input is

$$R(z) = \frac{1}{1 - z^{-1}}$$

we obtain

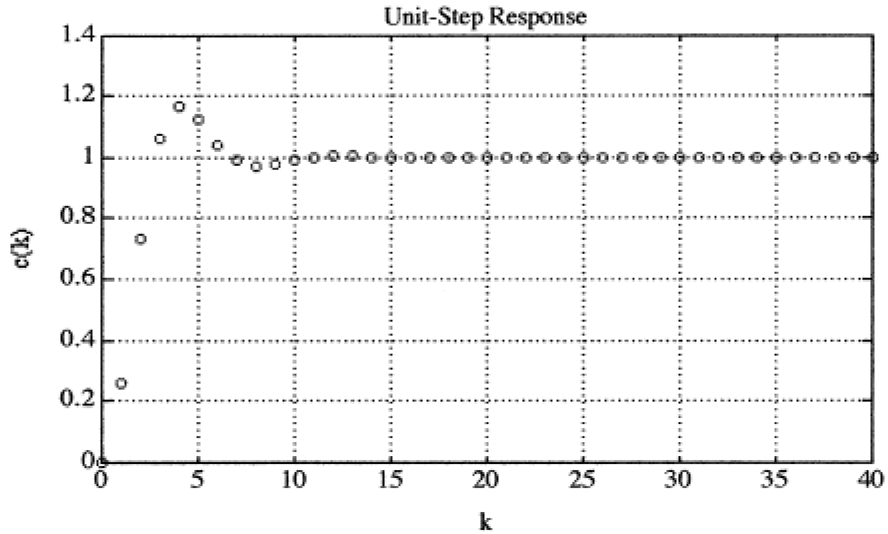
$$C(z) = \frac{0.2567z^{-1} + 0.2483z^{-2}}{(1 - 0.8987z^{-1} + 0.4037z^{-2})(1 - z^{-1})}$$

$$= \frac{0.2567z^{-1} + 0.2483z^{-2}}{1 - 1.8987z^{-1} + 1.3024z^{-2} - 0.4037z^{-3}}$$

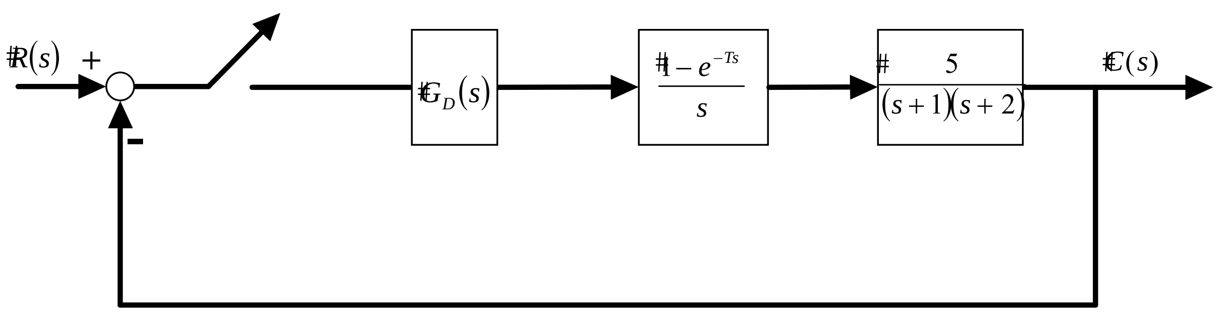
$$= 0.2567z^{-1} + 0.7357z^{-2} + 1.0625z^{-3} + 1.1629z^{-4}$$
$$+ 1.1212z^{-5} + 1.0431z^{-6} + 0.9898z^{-7} + 0.9735z^{-8}$$
$$+ 0.9803z^{-9} + 0.9930z^{-10} + 1.0017z^{-11} + 1.0043z^{-12} + \cdots$$

Computational solution with MATLAB:

```
»% MATLAB Program for Problem B-4-9
»
»% ----- Unit-step response -----
»
»num = [0   0.2567   0.2483];
»den = [1   -0.8987   0.4037];
»r = ones(1,41);
»v = [0   40   0   1.4];
»axis(v);
»k = 0:40;
»c = filter(num,den,r);
»plot(k,c,'o')
»grid
»title('Unit-Step Response')
»xlabel('k')
»ylabel('c(k)')
```



4) B-4-16 Consider the digital control system shown in figure. Using the bode diagram approach in the w plane, design a digital controller such that the phase margin is 60 degree, the gain margin is 12 dB or more, and the static velocity error constant is 5 sec $^{-1}$. The sampling period is assumed to be 0.1 sec.



For T = 0.1 sec, we have

$$G(z) = \mathcal{Z}\left[\frac{1 - e^{-Ts}}{s} \frac{5}{(s + 1)(s + 2)}\right] = (1 - z^{-1})\,\mathcal{Z}\left[\frac{5}{s(s + 1)(s + 2)}\right]$$

$$= \frac{0.02263(z + 0.9061)}{(z - 0.9048)(z - 0.8187)}$$

Using the transformation

$$z = \frac{1 + \frac{1}{2}Tw}{1 - \frac{1}{2}Tw} = \frac{1 + 0.05w}{1 - 0.05w}$$

we have

$$G(w) = \frac{0.02263\left(\frac{1 + 0.05w}{1 - 0.05w} + 0.9061\right)}{\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.9048\right)\left(\frac{1 + 0.05w}{1 - 0.05w} - 0.8187\right)}$$

$$= \frac{2.500\left(1 - \frac{1}{20}w\right)\left(1 + \frac{1}{406}w\right)}{(1 + w)\left(1 + \frac{1}{1.994}w\right)}$$

Notice that in order to have the static velocity error constant $K_v = 5\ \text{sec}^{-1}$, we need the controller $G_D(w)$ to include an integrator.

Using the conventional design approach, we find the following $G_D(w)$ will satisfy the requirements that the phase margin be 60°, the gain margin be not less than 12 db, and K_v be equal to $5\ \text{sec}^{-1}$.

$$G_D(w) = \frac{2}{w}\left(\frac{1 + \frac{1}{0.1}w}{1 + \frac{1}{0.01}w}\right)\left(\frac{1 + w}{1 + \frac{1}{10}w}\right)$$

Then the open-loop pulse transfer function becomes

$$G_D(w)G(w) = \frac{5}{w}\,\frac{\left(1 + \frac{1}{0.1}w\right)\left(1 - \frac{1}{20}w\right)\left(1 + \frac{1}{406}w\right)}{\left(1 + \frac{1}{0.01}w\right)\left(1 + \frac{1}{10}w\right)\left(1 + \frac{1}{1.994}w\right)}$$

From the Bode diagram of $G_D(w)G(w)$ (see next page), we find the phase margin to be approximately 60° and the gain margin to be approximately 22 dB. The gain crossover frequency is $\nu = 0.5$ rad/sec. The phase crossover frequency is $\nu = 3.5$ rad/sec.

Next, using the following transformation:

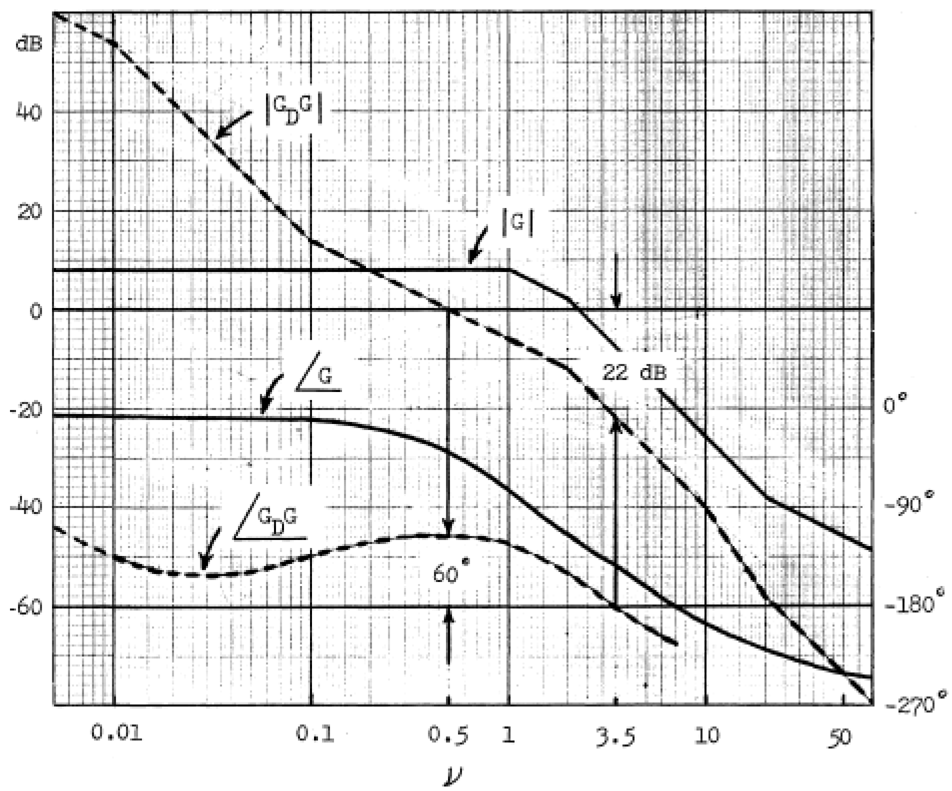
$$w = \frac{2}{0.1}\,\frac{z - 1}{z + 1} = 20\,\frac{z - 1}{z + 1}$$

we obtain $G_D(z)$ as follows:

$$G_D(z) = \frac{2\left(1 + \frac{1}{0.1}\,20\,\frac{z - 1}{z + 1}\right)\left(1 + 20\,\frac{z - 1}{z + 1}\right)}{20\left(\frac{z - 1}{z + 1}\right)\left(1 + \frac{1}{0.01}\,20\,\frac{z - 1}{z + 1}\right)\left(1 + \frac{1}{10}\,20\,\frac{z - 1}{z + 1}\right)}$$

$$= 0.07035\,\frac{(z + 1)(z - 0.9900)(z - 0.9048)}{(z - 1)(z - 0.9990)(z - 0.3333)}$$

The digital controller $G_D(z)$ defined by this last equation satisfies all the requirements of the problem and is, therefore, satisfactory.


dB
40
20
0
-20
-40
-60
|G_D G|
|G|
22 dB
∠G
∠G_D G
60°
0°
-90°
-180°
-270°
0.01
0.1
0.5
1
3.5
10
50
ν