II The Z Transform

Topics to be covered

1. Introduction
2. The Z transform
3. Z transforms of elementary functions
4. Properties and Theory of z transform
5. The inverse z transform
6. Z transform for solving difference equations

II.1 Introduction

The role of the z transform in discrete-time control system is similar to that of the Laplace transform in continuous-time system.
Discrete time signals: The sampled signal is $x(0), x(T), x(2T)\cdots$ where T is the sampling period. The signal can be written as $x(0), x(1), x(2)\cdots x(k)\cdots$ it can be considered as a sampled signal of $x(t)$ where T is 1 sec. $x(kT)$ and $x(k)$ are interchangeable if it doesn't make confusion.

II.2 The Z transform

In considering the z transform of a time function $x(t)$, we consider the sampled value of $x(t)$, that is $x(0), x(T), x(2T)\cdots$ The z transform of a time function , where t is nonnegative, is defined as follows

$$X(z) = Z[x(t)] = Z[x(kT)] = \sum_{k=0}^{+\infty} x(kT) z^{-k} \qquad 2.1$$

$$x(kT) \overset{Z}{\leftrightarrow} X(Z)$$

Definition: Z-transform of a general discrete-time signal $x(k)$ is defined as

$$X(Z) = \sum_{k=0}^{+\infty} x(k) z^{-k} \qquad 2.2$$

$$x(k) \overset{Z}{\leftrightarrow} X(Z)$$

Note: eq. 2.1 and 2.2 is referred to as the one-sided z transform or unilateral z transform. Z is a complex variable.

If $-\infty < t < \infty$, or $k = 0, \pm 1, \pm 2 \cdots$, then the z transform will be defined as

$$X(z) = Z[x(t)] = Z[x(kT)] = \sum_{k=-\infty}^{+\infty} x(kT) z^{-k} \qquad 2.3$$

$$x(kT) \overset{Z}{\leftrightarrow} X(Z)$$

Definition: Z-transform of a general discrete-time signal $x(k)$ is defined as

$$X(Z) = \sum_{k=-\infty}^{+\infty} x(k) z^{-k} \qquad 2.4$$

$$x(k) \overset{Z}{\leftrightarrow} X(Z)$$

Note: eq. 2.3 and 2.4 is referred to as the two-sided z transform or bilateral z transform. Z is a complex variable. We are only focused on one-sided z transform in this course.

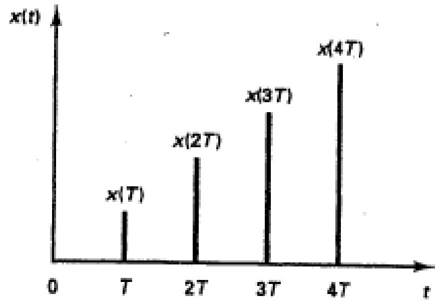
II.3 Z Transforms of elementary functions

Unit step function

$$x(t)=\begin{cases}1 & 0\le t\\ 0 & t<0\end{cases}$$

$$X(z)=Z[x(t)]=Z[x(kT)]=\sum_{k=0}^{+\infty}x(kT)z^{-k}=\sum_{k=0}^{+\infty}z^{-k}=\frac{1}{1-z^{-1}}=\frac{z}{z-1}\quad \text{(ROC: } |z|>1\text{)}$$

Unit ramp function



$$x(t)=\begin{cases}t & 0\le t\\ 0 & t<0\end{cases}$$

$$X(z)=Z[x(t)]=Z[x(kT)]=\sum_{k=0}^{+\infty}kTz^{-k}=T\sum_{k=0}^{+\infty}kz^{-k}=T\frac{z^{-1}}{(1-z^{-1})^2}=\frac{Tz}{(z-1)^2}$$

Polynomial function a^k

$$x(k)=\begin{cases}a^k & k=0,1,2\cdots\\ 0 & k<0\end{cases}$$

$$\begin{aligned}X(Z)&=\sum_{k=0}^{+\infty}x(k)z^{-k}\\ &=\sum_{k=0}^{+\infty}a^k z^{-k}\\ &=\sum_{k=0}^{+\infty}(az^{-1})^k\\ &=\frac{1}{1-az^{-1}}=\frac{z}{z-a}\end{aligned}$$

ROC: $|z|>|a|$

Exponential function

$$x(t)=\begin{cases} e^{-at} & 0 \le t \\ 0 & t<0 \end{cases}$$

$$\begin{aligned} X(Z) &= \sum_{k=0}^{+\infty} x(kT) z^{-k} \\ &= \sum_{k=0}^{+\infty} e^{-akT} z^{-k} \\ &= \sum_{k=0}^{+\infty} (e^{-aT} z^{-1})^{k} \\ &= \frac{1}{1-e^{-aT}z^{-1}} = \frac{z}{z-e^{-aT}} \end{aligned}$$

Example 2.1 Obtain the z transform of

$$x(t)=\begin{cases} e^{-at}\sin wt & 0 \le t \\ 0 & t<0 \end{cases}$$

$$e^{-at}\sin wt = e^{-at}\,\frac{1}{2j}\left(e^{jwt}-e^{-jwt}\right)=\frac{1}{2j}\left(e^{(-a+jw)t}-e^{(-a-jw)t}\right)$$

since

$$e^{-at} \overset{Z}{\leftrightarrow} \frac{z}{z-e^{-aT}} \Rightarrow e^{(-a+jw)t} \overset{Z}{\leftrightarrow} \frac{z}{z-e^{(-a+jw)T}},\, e^{(-a-jw)t} \overset{Z}{\leftrightarrow} \frac{z}{z-e^{(-a-jw)T}}$$

$$\Rightarrow e^{-at}\sin wt \overset{Z}{\leftrightarrow} \frac{1}{2j}\left(\frac{z}{z-e^{(-a+jw)T}}-\frac{z}{z-e^{(-a-jw)T}}\right)=\frac{e^{-aT}z^{-1}\sin wT}{1-2e^{-aT}z^{-1}\cos wT+e^{-2aT}z^{-2}}$$

Example 2.2 Obtain the z transform of

$$x(s)=\frac{1}{(s+a)^2}$$

$$x(s)=\frac{1}{(s+a)^2} \rightarrow x(t)=te^{-at}$$

$$X(Z) = \sum_{k=0}^{+\infty} x(kT) z^{-k}$$
$$= \sum_{k=0}^{+\infty} kTe^{-akT} z^{-k}$$
$$= T\sum_{k=0}^{+\infty} k(e^{-aT} z^{-1})^{k}$$
$$= \frac{Te^{-aT} z^{-1}}{(1 - e^{-aT} z^{-1})^{2}}$$

Table of z transforms 1

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
1.	—	—	Kronecker delta $\delta_0(k)$ $1, \quad k = 0$ $0, \quad k \neq 0$	1
2.	—	—	$\delta_0(n-k)$ $1, \quad n = k$ $0, \quad n \neq k$	z^{-k}
3.	$\frac{1}{s}$	$1(t)$	$1(k)$	$\frac{1}{1 - z^{-1}}$
4.	$\frac{1}{s+a}$	e^{-at}	e^{-akT}	$\frac{1}{1 - e^{-aT}z^{-1}}$
5.	$\frac{1}{s^2}$	t	kT	$\frac{Tz^{-1}}{(1 - z^{-1})^2}$
6.	$\frac{2}{s^3}$	t^2	$(kT)^2$	$\frac{T^2 z^{-1}(1 + z^{-1})}{(1 - z^{-1})^3}$
7.	$\frac{6}{s^4}$	t^3	$(kT)^3$	$\frac{T^3 z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^4}$
8.	$\frac{a}{s(s+a)}$	$1 - e^{-at}$	$1 - e^{-akT}$	$\frac{(1 - e^{-aT})z^{-1}}{(1 - z^{-1})(1 - e^{-aT}z^{-1})}$
9.	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at} - e^{-bt}$	$e^{-akT} - e^{-bkT}$	$\frac{(e^{-aT} - e^{-bT})z^{-1}}{(1 - e^{-aT}z^{-1})(1 - e^{-bT}z^{-1})}$
10.	$\frac{1}{(s+a)^2}$	te^{-at}	kTe^{-akT}	$\frac{Te^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$
11.	$\frac{s}{(s+a)^2}$	$(1 - at)e^{-at}$	$(1 - akT)e^{-akT}$	$\frac{1 - (1 + aT)e^{-aT}z^{-1}}{(1 - e^{-aT}z^{-1})^2}$

Table of z transforms 2

	$X(s)$	$x(t)$	$x(kT)$ or $x(k)$	$X(z)$
12.	$\frac{2}{(s+a)^3}$	$t^2 e^{-at}$	$(kT)^2 e^{-akT}$	$\frac{T^2 e^{-aT}(1+e^{-aT}z^{-1})z^{-1}}{(1-e^{-aT}z^{-1})^3}$
13.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$	$akT-1+e^{-akT}$	$\frac{[(aT-1+e^{-aT})+(1-e^{-aT}-aTe^{-aT})z^{-1}]z^{-1}}{(1-z^{-1})^2(1-e^{-aT}z^{-1})}$
14.	$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$	$\sin \omega kT$	$\frac{z^{-1}\sin \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
15.	$\frac{s}{s^2+\omega^2}$	$\cos \omega t$	$\cos \omega kT$	$\frac{1-z^{-1}\cos \omega T}{1-2z^{-1}\cos \omega T+z^{-2}}$
16.	$\frac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin \omega t$	$e^{-akT}\sin \omega kT$	$\frac{e^{-aT}z^{-1}\sin \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
17.	$\frac{s+a}{(s+a)^2+\omega^2}$	$e^{-at}\cos \omega t$	$e^{-akT}\cos \omega kT$	$\frac{1-e^{-aT}z^{-1}\cos \omega T}{1-2e^{-aT}z^{-1}\cos \omega T+e^{-2aT}z^{-2}}$
18.			a^k	$\frac{1}{1-az^{-1}}$
19.			a^{k-1} $k=1,2,3,\ldots$	$\frac{z^{-1}}{1-az^{-1}}$
20.			ka^{k-1}	$\frac{z^{-1}}{(1-az^{-1})^2}$
21.			$k^2 a^{k-1}$	$\frac{z^{-1}(1+az^{-1})}{(1-az^{-1})^3}$
22.			$k^3 a^{k-1}$	$\frac{z^{-1}(1+4az^{-1}+a^2z^{-2})}{(1-az^{-1})^4}$
23.			$k^4 a^{k-1}$	$\frac{z^{-1}(1+11az^{-1}+11a^2z^{-2}+a^3z^{-3})}{(1-az^{-1})^5}$
24.			$a^k \cos k\pi$	$\frac{1}{1+az^{-1}}$
25.			$\frac{k(k-1)}{2!}$	$\frac{z^{-2}}{(1-z^{-1})^3}$
26.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}$	$\frac{z^{-m+1}}{(1-z^{-1})^m}$
27.			$\frac{k(k-1)}{2!}a^{k-2}$	$\frac{z^{-2}}{(1-az^{-1})^3}$
28.			$\frac{k(k-1)\cdots(k-m+2)}{(m-1)!}a^{k-m+1}$	$\frac{z^{-m+1}}{(1-az^{-1})^m}$

II.4 Properties and Theory of z transform

i) Linearity

$$\left.\begin{aligned} x_1(k) &\leftrightarrow X_1(z), ROC = R_1 \\ x_2(k) &\leftrightarrow X_2(z), ROC = R_2 \end{aligned}\right\} \Rightarrow ax_1(k) + bx_2(k) \leftrightarrow aX_1(z) + bX_2(z), ROC = R_1 \cap R_2$$

ii) Scaling in the z-Domain

$$x(k) \leftrightarrow X(z), ROC = R \Rightarrow a^k x[k] \overset{Z}{\leftrightarrow} X(a^{-1}z), ROC = |a|R$$

Example 2.3 Determine the z transform and the associated region of convergence for following function of time: $u(k)$ and $a^k u(k)$

Since $u(k) \rightarrow \frac{1}{1-z^{-1}}$, we have $a^k u(k) \rightarrow \frac{1}{1-\left(a^{-1}z\right)^{-1}} = \frac{1}{1-az^{-1}}$

iii) Time Shifting

$$x(t-nT) \leftrightarrow z^{-n} X(z),$$
$$x(t+nT) \leftrightarrow z^n \left[X(z) - \sum_{k=0}^{n-1} x(kT) z^{-k} \right]$$

for number sequence k. we have

$$x(k-n) \leftrightarrow z^{-n} X(z),$$
$$x(k+n) \leftrightarrow z^n \left[X(z) - \sum_{k=0}^{n-1} x(k) z^{-k} \right]$$

Example 2.4. Determine the z transform for $u(k) - u(k-1)$

$$u(k) - u(k-1) \rightarrow \frac{1}{1-z^{-1}} - \frac{z^{-1}}{1-z^{-1}} = 1$$

Verify: $\delta(k) = u(k) - u(k-1) \rightarrow 1$

iv) Complex translation theorem

$$x(t) \leftrightarrow X(z) \Rightarrow e^{-at} x(t) \leftrightarrow X(ze^{aT})$$

v) Differentiation in the z-Domain

$$x(k) \leftrightarrow X(z), ROC = R \Rightarrow kx(k) \overset{Z}{\leftrightarrow} -z\frac{dX(z)}{dz}, ROC = R$$

vi) The initial value theorem

If $x(k) = 0, k < 0$, then $x(0) = \lim_{z\to\infty} X(z)$

Example2.5 Determine the z transform for following function of time: $x(k) = k\left(\frac{1}{2}\right)^k u(k)$

$$x_1(k) = \left(\frac{1}{2}\right)^k u(k) \to X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}, |z| > \frac{1}{2}$$

$x(k) = kx_1(k)$ Thus $X(z) = -z\frac{d}{dz}X_1(z) = -\frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2}, |z| > \frac{1}{2}$

verify the initial value theorem

$$x(0) = \lim_{z\to\infty} X(z) = \lim_{z\to\infty} -\frac{\frac{1}{2}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} = 0$$

vii) Final Value theorem

If $x(k) = 0, k < 0$, then $\lim_{k\to\infty} x(k) = \lim_{z\to 1}(1 - z^{-1})X(z)$

Example2.6 Determine the final value of $\frac{z}{z-e^{-aT}}$ by using the final value theorem.

$$\lim_{k\to\infty} x(k) = \lim_{z\to 1}(1 - z^{-1})X(z) = \lim_{z\to 1}(1 - z^{-1})\frac{z}{z-e^{-aT}} = 0$$

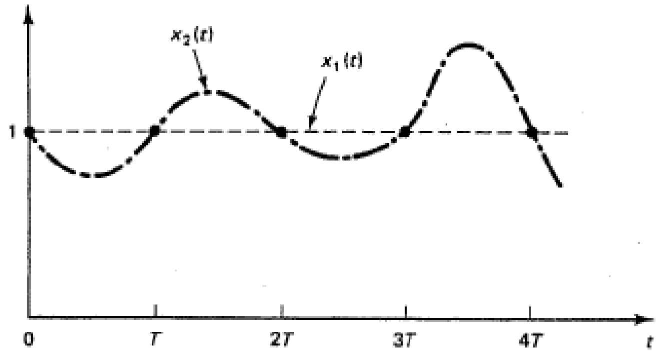
$x(t) = \begin{cases} e^{-at} & 0 \le t \\ 0 & t < 0 \end{cases}$ verify that $\lim_{k \to \infty} x(k) = 0$

Property table:

	$x(t)$ or $x(k)$	$\mathcal{Z}[x(t)]$ or $\mathcal{Z}[x(k)]$
1.	$ax(t)$	$aX(z)$
2.	$ax_1(t) + bx_2(t)$	$aX_1(z) + bX_2(z)$
3.	$x(t + T)$ or $x(k + 1)$	$zX(z) - zx(0)$
4.	$x(t + 2T)$	$z^2 X(z) - z^2 x(0) - zx(T)$
5.	$x(k + 2)$	$z^2 X(z) - z^2 x(0) - zx(1)$
6.	$x(t + kT)$	$z^k X(z) - z^k x(0) - z^{k-1} x(T) - \cdots - zx(kT - T)$
7.	$x(t - kT)$	$z^{-k} X(z)$
8.	$x(n + k)$	$z^k X(z) - z^k x(0) - z^{k-1} x(1) - \cdots - zx(k - 1)$
9.	$x(n - k)$	$z^{-k} X(z)$
10.	$tx(t)$	$-Tz \frac{d}{dz} X(z)$
11.	$kx(k)$	$-z \frac{d}{dz} X(z)$
12.	$e^{-at} x(t)$	$X(ze^{aT})$
13.	$e^{-ak} x(k)$	$X(ze^{a})$
14.	$a^k x(k)$	$X\left(\frac{z}{a}\right)$
15.	$ka^k x(k)$	$-z \frac{d}{dz} X\left(\frac{z}{a}\right)$
16.	$x(0)$	$\lim_{z \to \infty} X(z)$ if the limit exists
17.	$x(\infty)$	$\lim_{z \to 1} [(1 - z^{-1}) X(z)]$ if $(1 - z^{-1}) X(z)$ is analytic on and outside the unit circle
18.	$\nabla x(k) = x(k) - x(k - 1)$	$(1 - z^{-1}) X(z)$
19.	$\Delta x(k) = x(k + 1) - x(k)$	$(z - 1) X(z) - zx(0)$
20.	$\sum_{k=0}^{n} x(k)$	$\frac{1}{1 - z^{-1}} X(z)$
21.	$\frac{\partial}{\partial a} x(t, a)$	$\frac{\partial}{\partial a} X(z, a)$
22.	$k^m x(k)$	$\left(-z \frac{d}{dz}\right)^m X(z)$
23.	$\sum_{k=0}^{n} x(kT) y(nT - kT)$	$X(z) Y(z)$
24.	$\sum_{k=0}^{\infty} x(k)$	$X(1)$

II.5 The inverse z transform

Note: the inverse z transform yields the corresponding time sequence $x(k)$, but doesn't yield a unique $x(t)$



If the z transform is given as a ratio of two polynomials in z, then the inverse z transform may be obtained by several different methods, such as direct division method, the computational method, the partial-fraction-expansion method, and the inversion integral method.

1) direct division method

$$X(Z)=\sum_{k=0}^{+\infty} x(kT)z^{-k} = x(0)+x(T)z^{-1}+x(2T)z^{-2}+\cdots x(kT)z^{-k}+\cdots \text{ or}$$

$$X(Z)=\sum_{k=0}^{+\infty} x(k)z^{-k} = x(0)+x(1)z^{-1}+x(2)z^{-2}+\cdots x(k)z^{-k}+\cdots$$

Example2.7 Determine the inverse transform of $\dfrac{z^{-1}}{\left(1-2z^{-1}\right)^2}$

$$\begin{array}{l}
\qquad\qquad\qquad\qquad z^{-1}+4z^{-2}+12z^{-3}+32z^{-4}+80z^{-5}\cdots \\
1-4z^{-1}+4z^{-2}\ \big)\ z^{-1} \\
\qquad\qquad\qquad\qquad \underline{z^{-1}-4z^{-2}+4z^{-3}} \\
\qquad\qquad\qquad\qquad\qquad 4z^{-2}-4z^{-3} \\
\qquad\qquad\qquad\qquad\qquad \underline{4z^{-2}-16z^{-3}+16z^{-4}} \\
\qquad\qquad\qquad\qquad\qquad\qquad 12z^{-3}-16z^{-4} \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{12z^{-3}-48z^{-4}+48z^{-5}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad 32z^{-4}-48z^{-5} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{32z^{-4}-128z^{-5}+128z^{-6}} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 80z^{-5}-128z^{-6}
\end{array}$$

By observation, we can see $x(k)=k2^{k-1}$

2) The computational method

a) MATLAB approach

Example 2.8 find the inverse transform of $\dfrac{z^{-1}}{\left(1-2z^{-1}\right)^2}$

Let $G(z)=\dfrac{z^{-1}}{\left(1-2z^{-1}\right)^2}$, $X(z)=1$, $Y(z)=\dfrac{z^{-1}}{\left(1-2z^{-1}\right)^2}$ input $X(z)=1$ is the z transform of the Kronecker delta input. In MATLAB, the Kronecker delta input is given by $x=[1 \quad zeros(1,N)]$, where N corresponds to the end of the discrete time duration of the process considered.

```
% example 2.8
% enter the numerator and denominator
num=[0 1 0];
den=[1 -4 4];
% enter the Kronexker delta input and filter
x=[1 zeros(1,40)];
y=filter(num,den,x);
% plot the results
plot(k,y,'o');
grid
title('Response to Kronecker Delta input');
xlabel('k');
ylabel('y(k)');
```

b) Difference equation approach

$$G(z)=\frac{Y(z)}{X(z)}=\frac{z^{-1}}{\left(1-2z^{-1}\right)^2}\Rightarrow\left(1-4z^{-1}+4z^{-2}\right)Y(z)=z^{-1}X(z)$$

$$\Rightarrow y(k)-4y(k-1)+4y(k-2)=x(k-1)$$

For above difference equation, $x(0)=1$, $x(k)=0, k\neq 0$ $y(k)=0, k<0$

Thus $y(0)=0$

$$y(k)=4y(k-1)-4y(k-2)+x(k-1)$$

$$y(1)=4y(0)-4y(-1)+x(0)=1$$

$$y(2)=4y(1)-4y(0)+x(1)=4$$

$$y(3)=4y(2)-4y(1)=4*4-4=12$$

$$y(4)=4y(3)-4y(2)=4*12-4*4=32$$

$\vdots$

This iterative process can be easily programmed.

3) The partial-fraction-expansion method

$$X(z)=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z+b_m}{z^n+a_1z^{n-1}+\cdots a_{n-1}z+a_n}=\frac{b_0z^m+b_1z^{m-1}+\cdots+b_{m-1}z+b_m}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

Case 1 all pole are distinguished:

$$\frac{X(z)}{z}=\frac{a_1}{(z-p_1)}+\frac{a_2}{(z-p_2)}+\cdots\frac{a_n}{(z-p_n)},\ \text{where}\ a_i=\left[(z-p_i)\frac{X(z)}{z}\right]_{z=p_i}$$

Case 2 double pole $\frac{X(z)}{z}=\frac{c_1}{(z-p_1)^2}+\frac{c_2}{(z-p_1)}$

Then $c_1=\left[(z-p_1)^2\frac{X(z)}{z}\right]_{z=p_1}$, and $c_2=\left\{\frac{d}{dz}\left[(z-p_1)^2\frac{X(z)}{z}\right]\right\}_{z=p_1}$

Example 2.9 find the inverse transform of $\frac{1}{(z-1)(z-2)}$

$$X(z)=\frac{1}{(z-1)(z-2)}=\frac{1}{(z-2)}-\frac{1}{(z-1)}=\frac{z^{-1}}{(1-2z^{-1})}-\frac{z^{-1}}{(1-z^{-1})}$$

$$x(k)=2^{k-1}-u(k-1), k=1,2,3,\cdots$$

4) The inversion integral method.

$$x(kT)=x(k)=\frac{1}{2\pi j}\oint_C X(z)z^{k-1}dz$$

Where C is a circle with its center at the origin of the z plane such that all poles of $X(z)z^{k-1}$ are inside it.
Using complex variable theory, we have

$$x(kT)=x(k)=K_1+K_2+\cdots K_m$$
$$=\sum_{i=1}^{m} residue\ of\ X(z)z^{k-1}\ at\ pole\ z=z_i\ of\ X(z)z^{k-1}$$

Case 1: $X(z)z^{k-1}$ contains simple pole at $z=z_i$, $K=\lim_{z\to z_i}(z-z_i)X(z)z^{k-1}$

Case 2 $X(z)z^{k-1}$ contains a multiple pole of $z=z_i$ of order q.

$$K=\frac{1}{(q-1)!}\lim_{z\to z_i}\frac{d^{q-1}}{dz^{q-1}}(z-z_i)^q X(z)z^{k-1}$$

Example 2.10 Find the inverse transform of $\frac{1}{(z-1)(z-2)}$

Note that $X(z)z^{k-1} = \frac{z^{k-1}}{(z-1)(z-2)}$

For k=0, $X(z)z^{k-1}$ has three poles at 0,1,2. And for k=1,2,3,…, $X(z)z^{k-1}$ has two poles 1,2.

a) k=0. $x(k) = K_1 + K_2 + K_3$

$$K_1 = [\textit{residue at simple pole } z = 0]$$
$$= \lim_{z\to 0}(z-0)X(z)z^{0-1} = \lim_{z\to 0}\frac{1}{z-1}\frac{1}{z-2} = \frac{1}{2}$$

$$K_2 = [\textit{residue at simple pole } z = 1]$$
$$= \lim_{z\to 1}(z-1)X(z)z^{0-1} = \lim_{z\to 1}\frac{1}{z}\frac{1}{z-2} = -1$$

$$K_3 = [\textit{residue at simple pole } z = 2]$$
$$= \lim_{z\to 2}(z-2)X(z)z^{0-1} = \lim_{z\to 2}\frac{1}{z}\frac{1}{z-1} = \frac{1}{2}$$

Thus for k=0, $x(k) = K_1 + K_2 + K_3 = \frac{1}{2} - 1 + \frac{1}{2} = 0$

b) k=1,2,3…. $x(k) = K_1 + K_2$

$$K_1 = [\textit{residue at simple pole } z = 1]$$
$$= \lim_{z\to 1}(z-1)X(z)z^{k-1} = \lim_{z\to 1}\frac{1}{z-2}1^{k-1} = -1$$

$$K_2 = [\textit{residue at simple pole } z = 2]$$
$$= \lim_{z\to 2}(z-2)X(z)z^{k-1} = \lim_{z\to 2}\frac{1}{z-1}z^{k-1} = 2^{k-1}$$

$$x(k) = \begin{cases} 0 & k = 0 \\ 2^{k-1} - u(k-1) & k = 1,2,3,\cdots \end{cases}$$

Example 2.11 Find the inverse transform of $\dfrac{z^{-1}\left(1+az^{-1}\right)}{\left(1-az^{-1}\right)^3}$

$$X(z)z^{k-1}=\frac{z^{-1}\left(1+az^{-1}\right)}{\left(1-az^{-1}\right)^3}z^{k-1}=\frac{z^k(z+a)}{(z-a)^3}$$

For k=0,1,2,… $X(z)z^{k-1}$ has multiple pole a of order 3.

$$\begin{aligned}
K&=\frac{1}{(q-1)!}\lim_{z\to z_i}\frac{d^{q-1}}{dz^{q-1}}\left(z-z_i\right)^q X(z)z^{k-1}\\
&=\frac{1}{(3-1)!}\lim_{z\to a}\frac{d^{3-1}}{dz^{3-1}}(z-a)^3\frac{z^k(z+a)}{(z-a)^3}\\
&=\frac{1}{2}\lim_{z\to a}\frac{d^2}{dz^2}z^k(z+a)\\
&=k^2a^{k-1}
\end{aligned}$$

II.6 Z transform for solving difference equations

Note: difference equations can be solved using digital computer. However, closed form expressions cannot be obtained from the computer solution. Maple can do some simple ones.

Table: z transform of $x(k+m)$ **and** $x(k-m)$

Discrete function	z Transform
$x(k+4)$	$z^4X(z)-z^4x(0)-z^3x(1)-z^2x(2)-zx(3)$
$x(k+3)$	$z^3X(z)-z^3x(0)-z^2x(1)-zx(2)$
$x(k+2)$	$z^2X(z)-z^2x(0)-zx(1)$
$x(k+1)$	$zX(z)-zx(0)$
$x(k)$	$X(z)$
$x(k-1)$	$z^{-1}X(z)$
$x(k-2)$	$z^{-2}X(z)$
$x(k-3)$	$z^{-3}X(z)$
$x(k-4)$	$z^{-4}X(z)$

Example 2.12. For following difference equation and associated input and initial conditions, determine the zero-input and zero-state responses by using the z transform.

$$y(k)+3y(k-1)=x(k),$$

$$x(k)=\left(\frac{1}{2}\right)^k u(k)$$

$$y(-1)=1$$

Taking the unilateral z transform of both sides of the given difference equation, we get

$$Y(z)+3z^{-1}Y(z)+3y[-1]=X(z)$$

Setting $X(z)=0$, we get

$$Y(z)=\frac{-3}{1+3z^{-1}}$$

The inverse unilateral z transform gives the zero input response

$$y_{zi}(k)=-3(-3)^k u(k)$$

Since $x(k)=\left(\frac{1}{2}\right)^k u(k)$, we have

$$X(z)=\frac{1}{1-\frac{1}{2}z^{-1}},\ |z|>\frac{1}{2}$$

Setting $y[-1]=0$, we get

$$Y(z)+3z^{-1}Y(z)=\frac{1}{1-\frac{1}{2}z^{-1}}\Rightarrow Y(z)=\left(\frac{1}{1+3z^{-1}}\right)\left(\frac{1}{1-\frac{1}{2}z^{-1}}\right)=\left(\frac{6/7}{1+3z^{-1}}\right)+\left(\frac{1/7}{1-\frac{1}{2}z^{-1}}\right)$$

Zero state response: $y_{zs}(k)=\frac{6}{7}(-3)^k u(k)+\frac{1}{7}\left(\frac{1}{2}\right)^k u(k)$

Ex. B-2-2,B-2-5,B-2-8, B-2-13, B-2-17