III Z-Plane Analysis

Topics to be covered

1. Introduction
2. Impulse sampling and data hold
3. Obtaining the Z transforms by convolution
4. Signal reconstruction
5. The pulse transfer function
6. Digital controller and filters

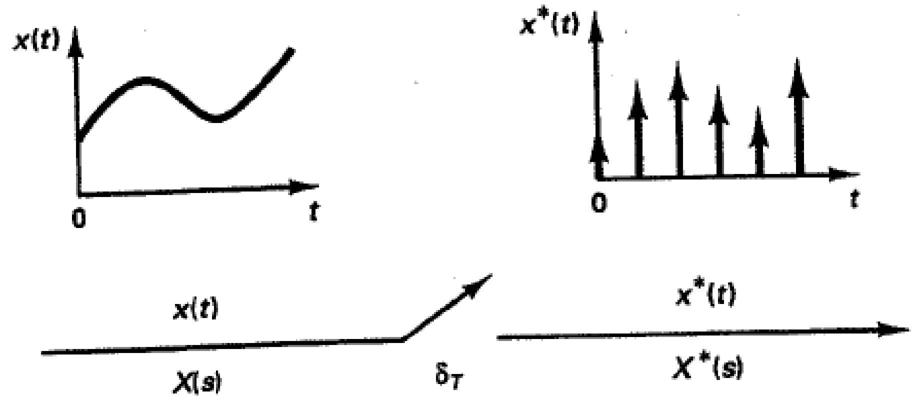
III.1 Introduction

The main advantage of the z transform method is that it enables the engineer to apply conventional continuous-time system design method to discrete-time systems.

Note: In this course, we assume the single sample rate and the sampling period is constant.

III.2 Impulse sampling and data hold

Impulse sampling:

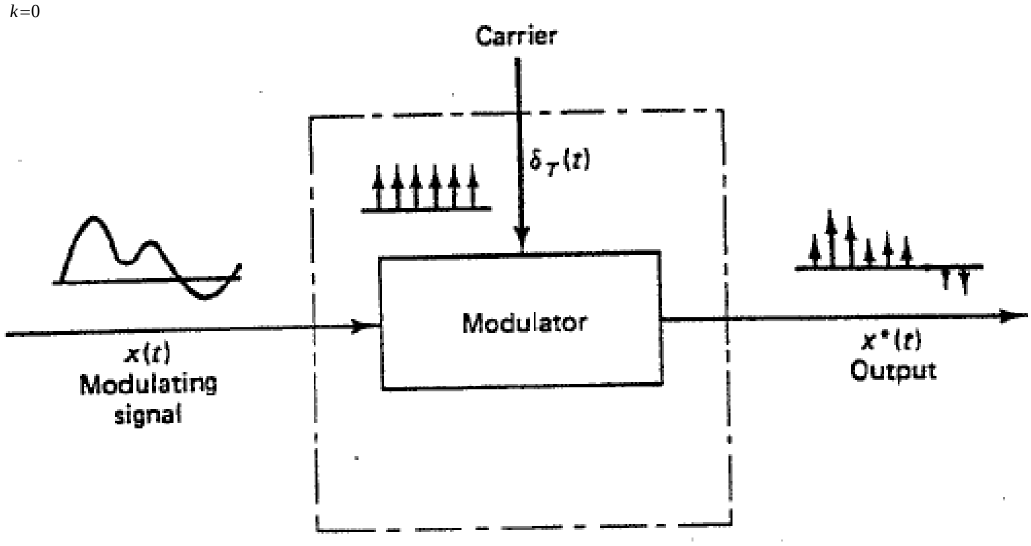


Note: input is continuous time signal $x(t)$. Output is a sequence of impulses, with the strength of each impulse equal to the magnitude of $x(t)$ at the corresponding instant of time. Sample signal $x*(t)$

$$x*(t)=\sum_{k=0}^{\infty}x(kT)\delta(t-kT) \qquad 3.1$$
$$=x(0)\delta(t)+x(T)\delta(t-T)+\cdots+x(kT)\delta(t-kT)+\cdots$$

Define a train of unit impulses as $\delta_T(t)$, or

$$\delta_T(t)=\sum_{k=0}^{\infty}\delta(t-kT)$$



$x(t)$ can be treated as the modulating signal, and $\delta_T(t)$ can be treated as the carrier.

Next find the Laplace transform of $x*(t)$

$$X*(s) = \int_{-\infty}^{\infty} x*(t)e^{-st}dt$$

$$= \int_{-\infty}^{\infty}\sum_{k=0}^{\infty} x(kT)\delta(t-kT)e^{-st}dt$$

$$= \sum_{k=0}^{\infty} x(kT)\left\{\int_{-\infty}^{\infty}\delta(t-kT)e^{-st}dt\right\} \qquad 3.2$$

$$= \sum_{k=0}^{\infty} x(kT)e^{-kTs}$$

If we define $e^{Ts} = z$ or $s = \frac{1}{T}\ln z$

Then 3.2 becomes:

$$X*(s)\Big|_{s=\frac{1}{T}\ln z} = \sum_{k=0}^{+\infty} x(kT)z^{-k} = X(z) \qquad 3.3$$

Data hold circuit:

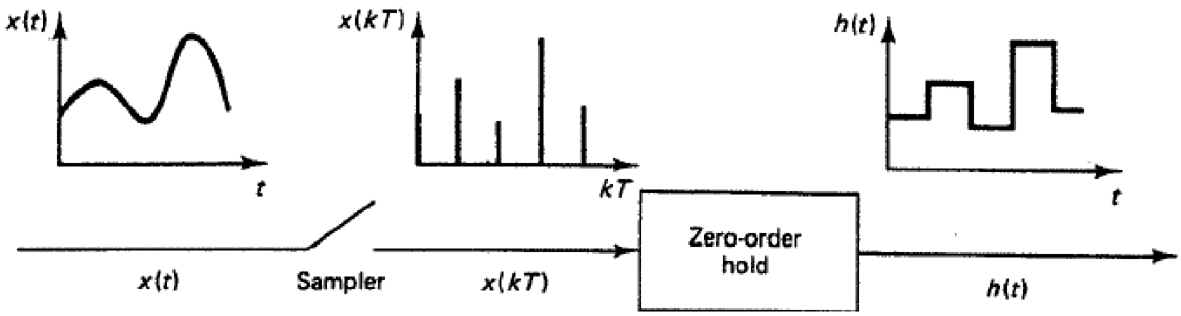
Data hold generates a continuous-time signal $h(t)$ from a discrete-time sequence $x(kT)$ $h(t)$ can be approximated by a polynomial in τ, where $0 \le \tau < T$, also $h(kT) = x(kT)$
We can have:

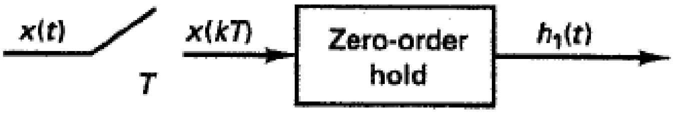
$$h(kT+\tau) = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_1\tau + x(kT) \qquad 3.4$$

If the data-hold circuit is an nth order polynomial extrapolator, it is called an nth-order hold.

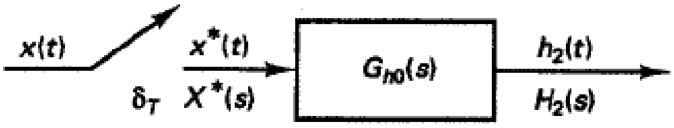
Note: Higher the order, longer the delay.

Zero order hold when n=0. Which means $h(kT+\tau) = x(kT)$





(a)



To obtain the mathematical model of the real sampler and zero-order circuit:

$$h_1(t)=\sum_{k=0}^{\infty}x(kT)[u(t-kT)-u(t-(k+1)T)]$$

since $u(t-kT)\rightarrow \frac{e^{-kTs}}{s}$

$$H_1(s)=\sum_{k=0}^{\infty}x(kT)\left[\frac{e^{-kTs}-e^{-(k+1)Ts}}{s}\right]$$
$$=\frac{1-e^{-Ts}}{s}\sum_{k=0}^{\infty}x(kT)e^{-kTs} \qquad 3.5$$

$$H_2(s)=H_1(s) \qquad 3.6$$

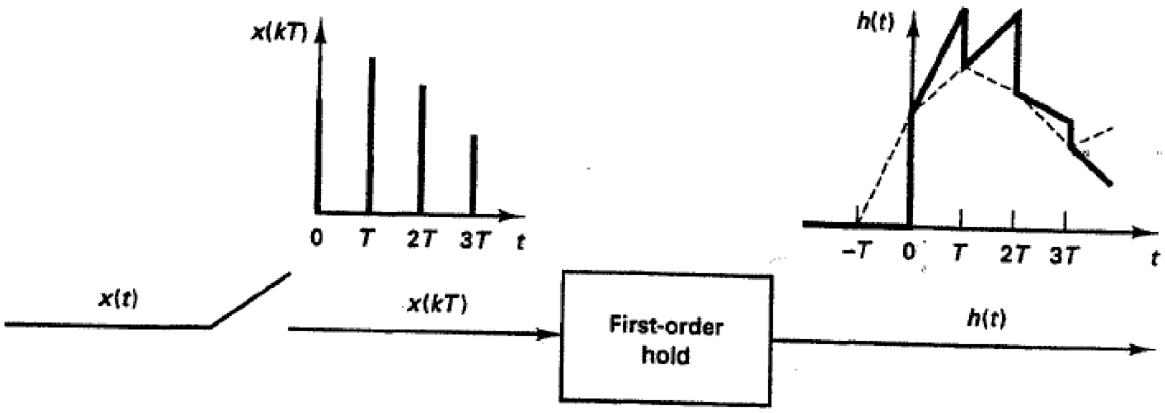
$$H_2(s)=G_{h0}(s)X*(s)=G_{h0}(s)\sum_{k=0}^{\infty}x(kT)e^{-kTs} \qquad 3.7$$

From 3.5-3.7 we have $G_{h0}(s)=\frac{1-e^{-Ts}}{s}$

Transfer function of first order hold.

$$h(kT+\tau)=a_1\tau+x(kT)$$
$$h(kT-T)=-a_1T+x(kT)=x((k-1)T)\rightarrow a_1=\frac{x(kT)-x((k-1)T)}{T}$$

$$h(kT+\tau)=\frac{x(kT)-x((k-1)T)}{T}\tau+x(kT)$$



$$X^*(s) = \sum_{k=0}^{\infty} x(kT) e^{-kTs}$$

$$h_1(t) = \sum_{k=0}^{\infty} \left(x(kT) + \frac{x(kT) - x((k-1)T)}{T}(t - kT) \right) [u(t - kT) - u(t - (k+1)T)]$$

Repeat the procedure of 3.5-7

$$G_{h1}(s) = \left(\frac{1 - e^{-Ts}}{s} \right)^2 \frac{Ts + 1}{T}$$

Note:

1) impulse sampling assumes the sampling duration equal to zero
2) many continuous time technique can be applied to discrete time system

III.3 Obtaining the Z transforms by convolution

$$x^*(t) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)$$

$$= x(t)\sum_{k=0}^{\infty} \delta(t-kT)$$

$$L[\delta(t-kT)] = e^{-kTs}$$

$$\Rightarrow L\left[\sum_{k=0}^{\infty} \delta(t-kT)\right] = \sum_{k=0}^{\infty} e^{-kTs} = \frac{1}{1-e^{-Ts}}$$

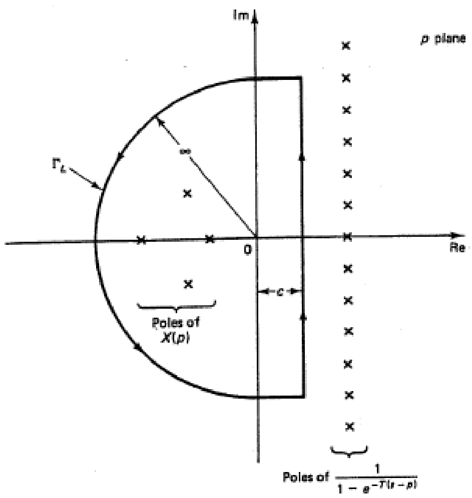
$$X^*(s) = L\left[x(t)\sum_{k=0}^{\infty} \delta(t-kT)\right]$$

Laplace transform of the product of two time functions $f(t)$ and $g(t)$
Can be given by:

$$L[f(t)g(t)] = \int_0^{\infty} f(t)g(t)e^{-st}dt = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p)G(s-p)dp \,,$$

where $F(p)$ is the LapLace transform of $f(t)$, and $G(p)$ is the transform of $g(t)$
So we can determine $X^*(s)$

$$X^*(s) = L\left[x(t)\sum_{k=0}^{\infty} \delta(t-kT)\right] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(p)\frac{1}{1-e^{-T(s-p)}}dp$$



note: The integration line is from $c - j\infty$ to $c + j\infty$ and this line is parallel to the imaginary axis in the p plane and separates the poles of $X(p)$ from those of $\frac{1}{1-e^{-T(s-p)}}$.

$$X^*(s) = L\left[x(t)\sum_{k=0}^{\infty}\delta(t-kT)\right] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} X(p)\frac{1}{1-e^{-T(s-p)}}dp$$

$$= \frac{1}{2\pi j}\oint X(p)\frac{1}{1-e^{-T(s-p)}}dp - \frac{1}{2\pi j}\int_{\Gamma} X(p)\frac{1}{1-e^{-T(s-p)}}dp$$

3.8

Where Γ is the semicircle of infinite radius in the left or right half p plane.

Case a: Evaluation of the convolution integral in the left half plane.

assume $X(s) = \frac{q(s)}{P(s)}$, where $q(s)$ and $p(s)$ are polynomials in s. We also assume $p(s)$ is of a higher order degree in s than $q(s)$, which means that $\lim_{s\to\infty} X(s) = 0$.
equation 3.8 will have one item left, the other one is reduced to zero

$$X^*(s) = \frac{1}{2\pi j}\oint X(p)\frac{1}{1-e^{-T(s-p)}}dp \qquad 3.9$$

3.9 is equal to the sum of the residue of $X(p)$ in the closed contour.

$$X^*(s) = \sum residue\ of\ \frac{X(p)}{1-e^{-T(s-p)}} at\ pole\ of\ X(p) \qquad 3.10$$

by substituting z for e^{Ts}, we have

$$X(z) = \sum residue\ of\ \frac{X(p)}{1-z^{-1}e^{Tp}} at\ pole\ of\ X(p)$$

by changing the notation from p to s, we obtain

$$X(z) = \sum residue\ of\ \frac{X(s)}{1-z^{-1}e^{Ts}} at\ pole\ of\ X(s)$$

Assume $X(s)$ has simple poles, $s_1, s_2, \cdots s_m$, corresponding K_j

$$K_j = \lim_{s\to s_j}\left[(s-s_j)\frac{X(s)z}{z-e^{Ts}}\right]$$

if pole at $s = s_j$ is a multiple pole of order n_i, then the residue K_j

$$K_j = \frac{1}{(n_i-1)!}\lim_{s\to s_j}\frac{d^{n_i-1}}{ds^{n_i-1}}\left[(s-s_j)^{n_i}\frac{X(s)z}{z-e^{Ts}}\right]$$

Example 3.1 obtain the z transform of $X(s)=\dfrac{1}{s(s+1)}$

We have simple pole at $s=0$ and $s=-1$

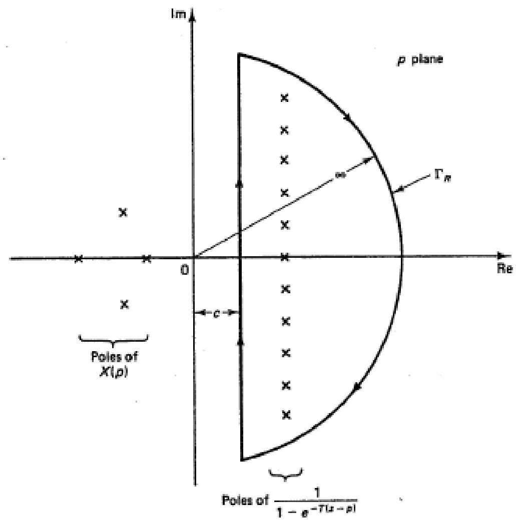
$$X(z)=\sum residue\ of\ \frac{X(s)}{1-z^{-1}e^{Ts}}\ at\ pole\ of\ X(s)$$

$$=\lim_{s\to 0}\left[(s-0)\frac{z}{z-e^{Ts}}\frac{1}{s(s+1)}\right]+\lim_{s\to -1}\left[(s+1)\frac{z}{z-e^{Ts}}\frac{1}{s(s+1)}\right]$$

$$=\frac{z}{z-1}-\frac{z}{z-e^{-T}}=\frac{(1-e^{-T})z}{(z-1)(z-e^{-T})}$$

Case b: evaluation of the convolution integral in the right half plane.

The closed contour encloses all poles of $\dfrac{1}{1-e^{-T(s-p)}}$, but it doesn't enclosed any poles of $X(p)$.



$$X*(s)=\frac{1}{2\pi j}\oint X(p)\frac{1}{1-e^{-T(s-p)}}dp-\frac{1}{2\pi j}\int_{\Gamma_R}X(p)\frac{1}{1-e^{-T(s-p)}}dp \qquad 3.11$$

Case 1: $X(s)$ has denominator two or more degrees higher in s than the numerator.

$$\lim_{s\to\infty}sX(s)=x(0+)=0\Rightarrow\int_{\Gamma_R}X(p)\frac{1}{1-e^{-T(s-p)}}dp=0$$

Thus $X^*(s) = \frac{1}{2\pi j}\oint X(p)\frac{1}{1-e^{-T(s-p)}}dp = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(s+jw_s k)$ (see A-3-7) 3.12

thus $X(z) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(s+jw_s k)\Big|_{s=\frac{1}{T}\ln z}$

Case 2: $X(s)$ has denominator one degrees higher in s than the numerator.

$$\int_{\Gamma_R} X(p)\frac{1}{1-e^{-T(s-p)}}dp = -\frac{1}{2}x(0+)$$

$$X^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(s+jw_s k) + \frac{1}{2}x(0+) \qquad 3.13$$

Obtaining the z transforms of functions involving the term $\left(1-e^{-Ts}\right)/s$

a) zero hold:

$$X(s) = \frac{\left(1-e^{-Ts}\right)}{s}G(s) \qquad 3.14$$

$$\Rightarrow X(z) = \left(1-z^{-1}\right)Z\left[\frac{G(s)}{s}\right]$$

b) first order hold:

$$X(s) = \left(\frac{\left(1-e^{-Ts}\right)}{s}\right)^2 \frac{Ts+1}{T}G(s) \qquad 3.15$$

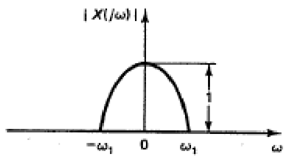
$$\Rightarrow X(z) = \left(1-z^{-1}\right)^2 Z\left[\frac{Ts+1}{Ts^2}G(s)\right]$$

Example 3.2 obtain the z transform of $X(s) = \frac{1-e^{-Ts}}{s^2}$

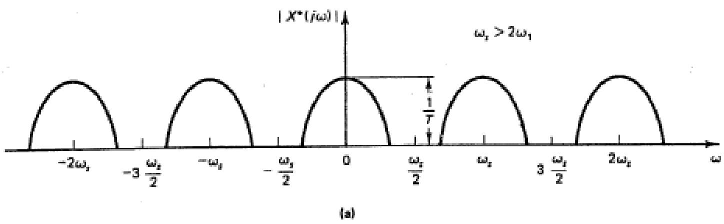
$$X(z) = \left(1-z^{-1}\right)Z\left[\frac{1}{s^2}\right] = \left(1-z^{-1}\right)\frac{Tz^{-1}}{\left(1-z^{-1}\right)^2} = \frac{Tz^{-1}}{\left(1-z^{-1}\right)}$$

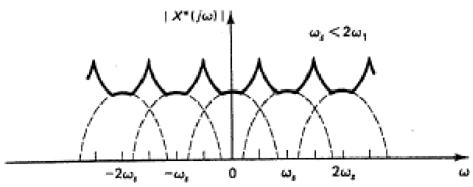
III.4 Signal Reconstruction

Sampling theorem: Assume the signal $x(t)$ has a frequency spectrum shown below. The signal doesn't contain any frequency components above w_1 radians per second. If w_s, defined as $\frac{2\pi}{T}$ where T is the sampling period, is greater than $2w_1$ or $w_s > 2w_1$, then the signal $x(t)$ can be reconstructed completely from the sampled signal $x^*(t)$



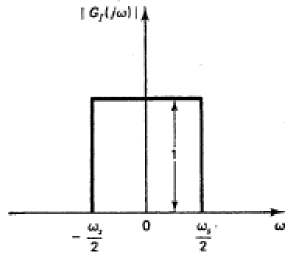
Recall 3.12,13. $X(z) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(s + jw_s k)\Big|_{s=\frac{1}{T}\ln z}$ or $X^*(s) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(s + jw_s k) - \frac{1}{2}x(0+)$



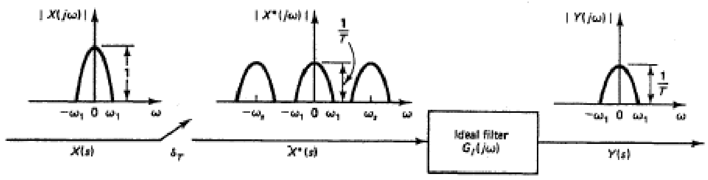


Two cases presented in the plot. $w_s > 2w_1$ and $w_s < 2w_1$

Ideal Low pass filter: unity over the frequency range: $-\frac{w_s}{2} < w < +\frac{w_s}{2}$ and zero outside the frequency range.

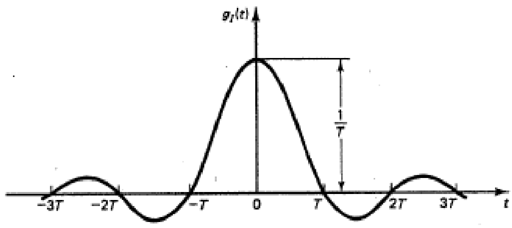


Following figures show the frequency spectra of the signals before and after ideal filtering.



It is shown that when $w_1 < \frac{w_s}{2} \Rightarrow 2w_1 < w_s$ $x(t)$ can be reconstructed completely from the sampled signal $x^*(t)$.

Ideal Low-pass filter is not physically realizable.



$$G_I(w) = \begin{cases} 1 & -\frac{1}{2}w_s \le w \le \frac{1}{2}w_s \\ 0 & elsewhere \end{cases} \Rightarrow g_i(t) = \frac{1}{T}\frac{\sin(w_s t/2)}{w_s t/2} \qquad 3.16$$

Note:

1) **3.16 gives the unit-impulse response of the ideal filter.** $-\infty < t < \infty$. It is impractical to have the signal before $t = 0$.
2) **In communication system, phase lag has been added so recover the signal. It is not desirable in control system, phase lag may make the system unstable.**
3) **It is practically impossible to reconstruct exactly a continuous time signal once it is sampled.**

Frequency response characteristics of the zero-order hold:

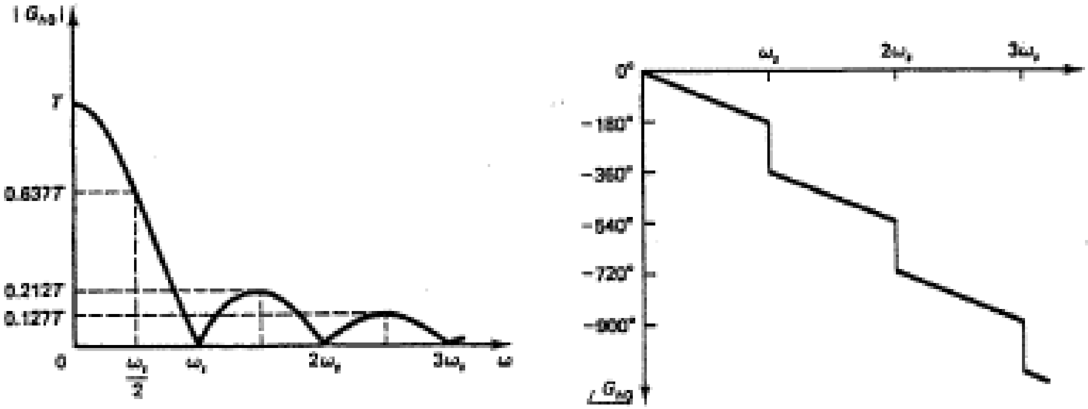
$$G_{h0}(s)=\frac{1-e^{-Ts}}{s}\Rightarrow G_{h0}(jw)=\frac{1-e^{-jTw}}{jw}=T\,\frac{\sin(wT/2)}{wT/2}e^{-(1/2)Tjw}$$

Magnitude: Becomes zero at the frequency equal to the sampling frequency and integral multiples of the sampling frequency.

$$\left|G_{h0}(jw)\right| == T\,\frac{\sin(wT/2)}{wT/2}$$

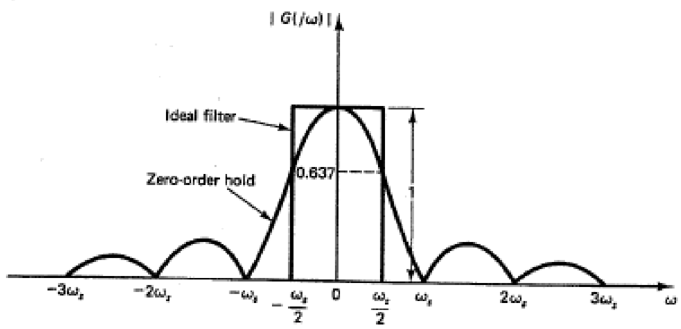
Note: Since the magnitude is not constant, distortion of the frequency spectra occurs in the system.

Phase: $\angle G_{h0}(jw)=\angle\frac{\sin(wT/2)}{wT/2}+\angle e^{-(1/2)jwT}=\angle\sin(wT/2)-\frac{wT}{2}$



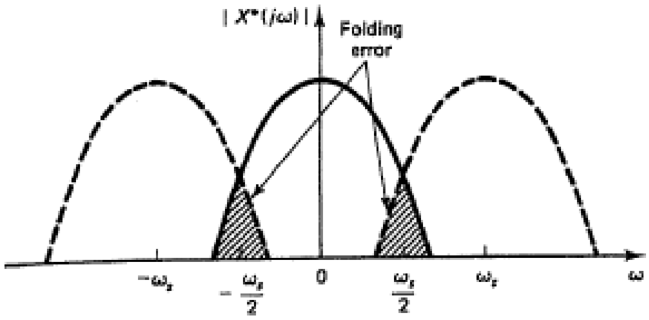
Magnitude and phase plot of the zero order hold

Following figure compares the ideal low pass filter with the zero-order hold.



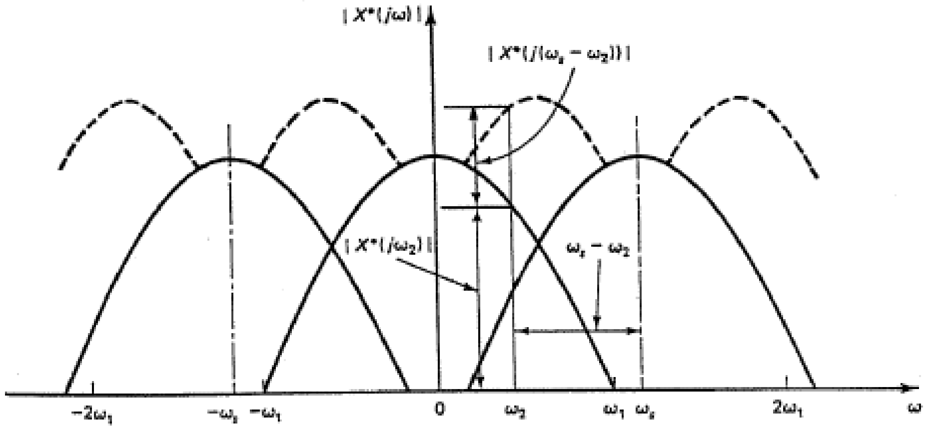
Folding: The phenomenon of the overlap in the frequency spectra is known as folding.

$\frac{w_s}{2}$ is called the folding frequency or Nyquist frequency $w_N = w_s / 2 = \frac{\pi}{T}$



Aliasing:

Consider the situation $w_s < 2w_1$. Consider an arbitrary frequency point w_2 that falls in the region of the overlap of the frequency spectra. The frequency spectrum at $w = w_2$ comprises two components, $|X*(jw_2)|$ and $|X*(j(w_s - w_2))|$. The latter component comes from the frequency spectrum centered at $w = w_s$ It is not possible to distinguish the frequency spectrum at $w = w_2$ from that at $w = nw_s - w_2$. The phenomenon that the frequency component $w = nw_s - w_2$ shows up at frequency w_2 when the signal is sampled is called aliasing.



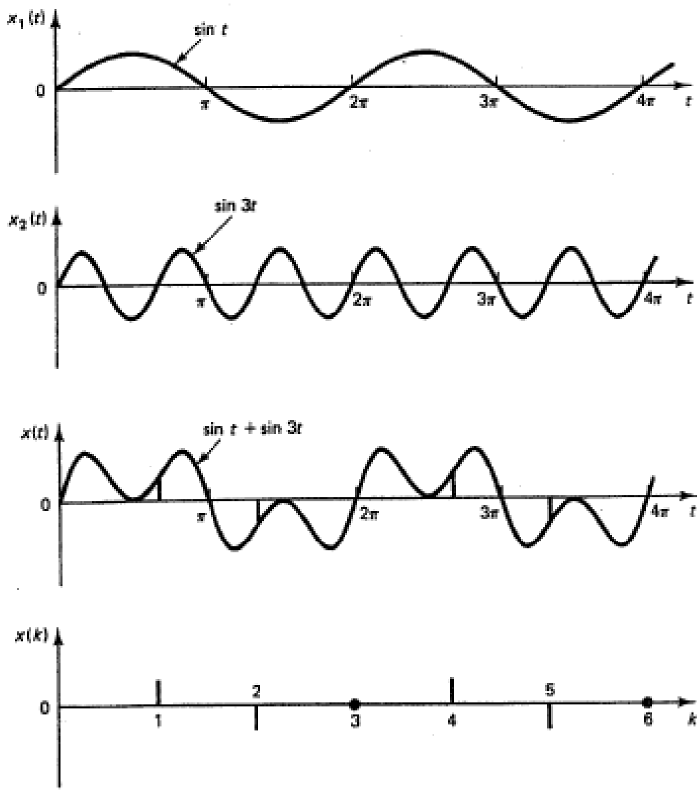
Note: to avoid aliasing, we either have $w_s > 2w_1$, or we have prefilter to ahead to reshape the frequency spectrum of the signal before the signal is sampled.

Hidden Oscillation:

It is noted that, if the continuous time signal $x(t)$ **involves a frequency component equal to** n **(integer) times the sampling frequency** w_s, then that component may not appear in the sampled signal.

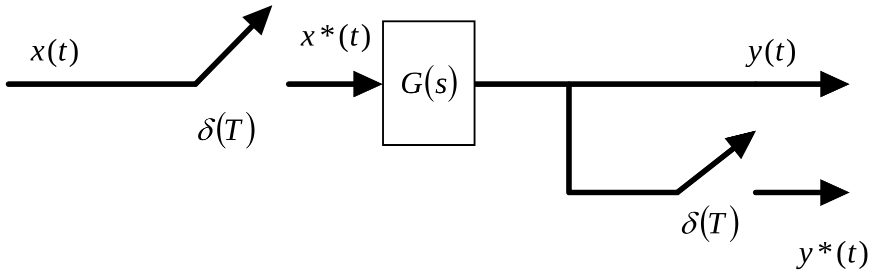
following figure shows that $x(t) = \sin t + \sin 3t$ is sampled at $t = 0, \frac{2\pi}{3}, \frac{4\pi}{3} \cdots$ ($w_s = 3rad/\sec$) $\sin 3t$ disappear, this is called hidden oscillation.

Note: reason: sample frequency is same as $\sin 3t$, and the sample point at $\sin 3t = 0$ Only way to detect this is through increased sampling rate or variable sampling rate.



III.5 The pulse transfer function

The pulse transfer function relates the z transform of the output at the sampling instants to that of the sampled input.



z transform of $y(t)$ is $Z[y(t)] = Y(z) = \sum_{k=0}^{\infty} y(kT) z^{-k}$ 3.17

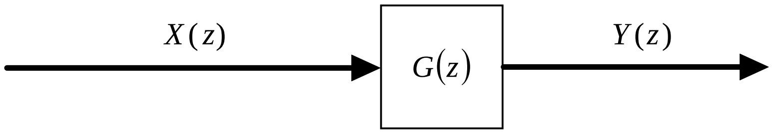
z transform of $y*(t)$ can be also given by 3.17

$$x*(t) = \sum_{k=0}^{\infty} x(t)\delta(t-kT) = \sum_{k=0}^{\infty} x(kT)\delta(t-kT)$$

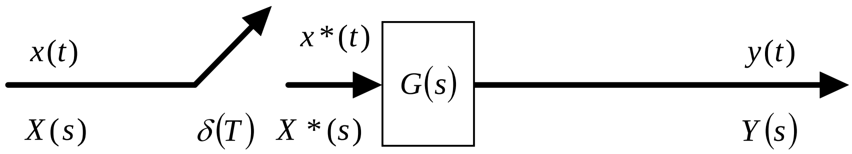
$$y(kT) = \sum_{h=0}^{k} x(kT-hT)g(hT) = \sum_{h=0}^{k} g(kT-hT)x(hT) = x(kT) * g(KT)$$

$$
\begin{aligned}
Z[y(t)] = Y(z) &= \sum_{k=0}^{\infty} y(kT) z^{-k} \\
&= \sum_{k=0}^{\infty} y(kT) z^{-k} \\
&= \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} x(kT-hT) g(hT) z^{-k} \\
&= \sum_{m=0}^{\infty} \sum_{h=0}^{\infty} x(mT) g(hT) z^{-(m+h)} \\
&= \sum_{m=0}^{\infty} x(mT) z^{-m} \sum_{h=0}^{\infty} g(hT) z^{-h} \\
&= G(z) X(z)
\end{aligned}
$$

$G(z)=\dfrac{Y(z)}{X(z)}$, the ratio of the pulsed output $Y(z)$ and the pulsed input $X(z)$ is called the pulse transfer function of the discrete-time system.



Starred Laplace transform of the signal involving both ordinary and starred Laplace transforms.



$Y(s)=G(s)X^*(s)$

We show that $Y^*(s)=G^*(s)X^*(s)$

$$y(t)=g(t)*x^*(t)$$

$$=\int_0^t g(t-\tau)x^*(\tau)d\tau$$

proof:

$$=\int_0^t g(t-\tau)\sum_{k=0}^{\infty}x(\tau)\delta(\tau-kT)d\tau$$

$$=\sum_{k=0}^{\infty}x(kT)g(t-kT)$$

$$Z[y(t)] = Y(z) = \sum_{k=0}^{\infty} y(kT)z^{-k}$$

$$= \sum_{k=0}^{\infty} y(kT)z^{-k}$$

$$= \sum_{k=0}^{\infty}\sum_{h=0}^{\infty} x(kT - hT)g(hT)z^{-k} \qquad 3.18$$

$$= \sum_{m=0}^{\infty}\sum_{h=0}^{\infty} x(mT)g(hT)z^{-(m+h)}$$

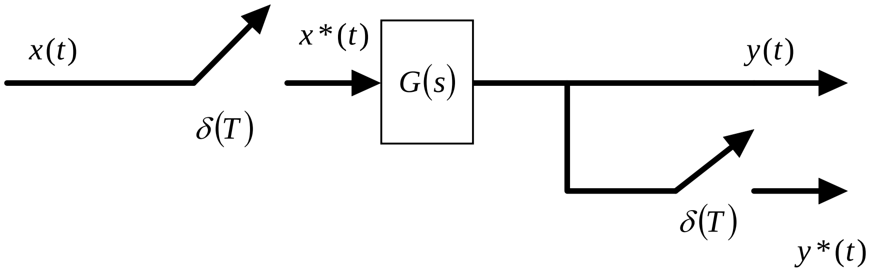
$$= \sum_{m=0}^{\infty} x(mT)z^{-m}\sum_{h=0}^{\infty} g(hT)z^{-h}$$

$$= G(z)X(z)$$

3.18 can be expressed as $Y^*(s)=G^*(s)X^*(s)$

Note: Taking the Laplace transform of a product of transforms, where some are ordinary Laplace transforms and others are starred Laplace transforms, the functions already in starred transforms can be factored out of the starred Laplace transform operation.

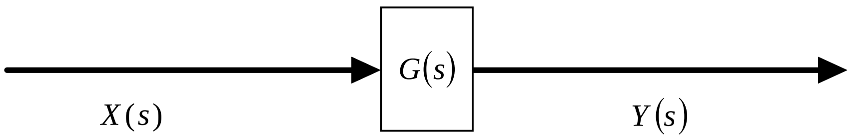
General procedure of obtaining pulse transfer function:



$Y(s) = G(s)X^*(s)$

$Y^*(s) = G^*(s)X^*(s)$

$G(z) = \dfrac{Y(z)}{X(z)}$



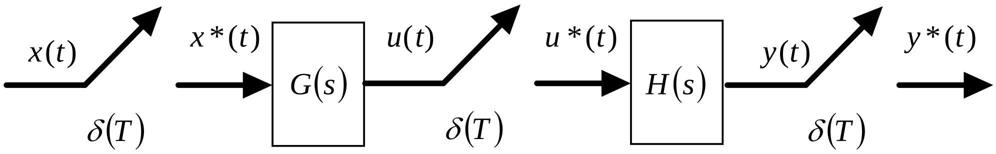
$Y(s) = G(s)X(s)$

$Y^*(s) = [G(s)X(s)]^* = [GX(s)]^*$

Example 3.3 obtain the z transform of $X(s) = \dfrac{1 - e^{-Ts}}{s(s+1)}$

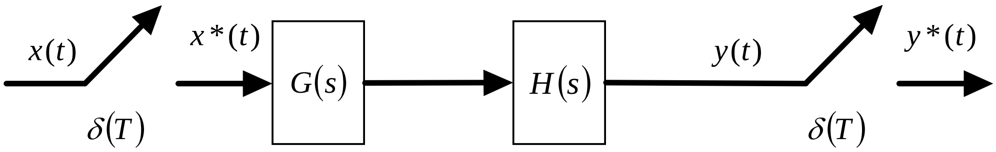
$$X(z) = (1 - z^{-1})Z\left[\frac{1}{s(s+1)}\right] = (1 - z^{-1})\left(\frac{1}{(1 - z^{-1})} - \frac{1}{(1 - e^{-T}z^{-1})}\right) = \frac{1 - e^{-T}z^{-1} - 1 + z^{-1}}{(1 - e^{-T}z^{-1})} = \frac{z^{-1}(1 - e^{-T})}{(1 - e^{-T}z^{-1})}$$

Pulse Transfer function of Cascaded Elements:



(a)

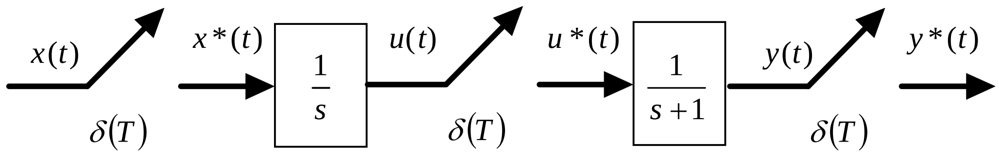
$U(s) = G(s)X^*(s) \Rightarrow U^*(s) = G^*(s)X^*(s)$
$Y(s) = H(s)U^*(s) \Rightarrow Y^*(s) = H^*(s)U^*(s) = H^*(s)G^*(s)X^*(s)$
Thus $Y(z) = H(z)G(z)X(z) \Rightarrow \dfrac{Y(z)}{X(z)} = H(z)G(z)$



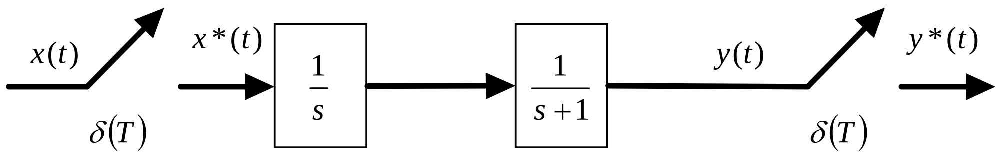
$Y(s) = G(s)H(s)X^*(s) = GH(s)X^*(s)$, where $GH(s) = G(s)H(s)$
$Y^*(s) = [GH(s)]^* X^*(s)$

Thus $Y(z) = GH(z)X(z) \Rightarrow \dfrac{Y(z)}{X(z)} = GH(z)$, note : $GH(z) \neq G(z)H(z)$

Example 3.4 Obtain the z transform of following two configuration:



(a)



(b)

for a)

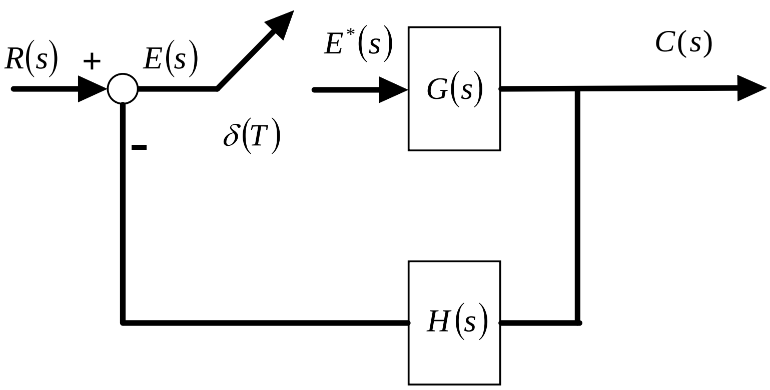
$$\frac{Y(z)}{X(z)} = H(z)G(z) = Z\left[\frac{1}{s}\right]Z\left[\frac{1}{s+1}\right] = \frac{1}{\left(1-z^{-1}\right)}\frac{1}{\left(1-e^{-T}z^{-1}\right)}$$

for b)

$$\frac{Y(z)}{X(z)} = GH(z) = Z\left[\frac{1}{s}\frac{1}{s+1}\right] = Z\left[\frac{1}{s}\right] - Z\left[\frac{1}{s+1}\right] = \frac{1}{1-z^{-1}} - \frac{1}{1-e^{-T}z^{-1}} = \frac{-e^{-T}z^{-1}+z^{-1}}{\left(1-z^{-1}\right)\left(1-e^{-T}z^{-1}\right)}$$

Apparent that $GH(z) \neq G(z)H(z)$

Pulse Transfer function of closed loop system:



$E(s) = R(s) - H(s)C(s)$

$C(s) = G(s)E^*(s) \Rightarrow C^*(s) = G^*(s)E^*(s)$

hence $E(s) = R(s) - H(s)C(s) = R(s) - H(s)G(s)E^*(s)$

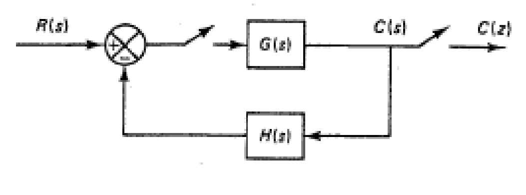
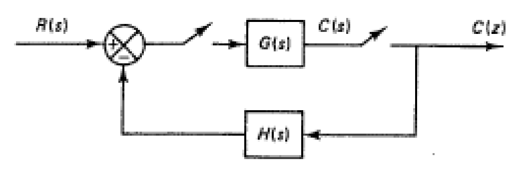
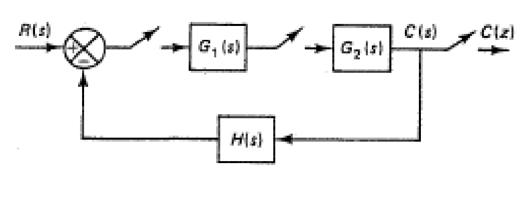
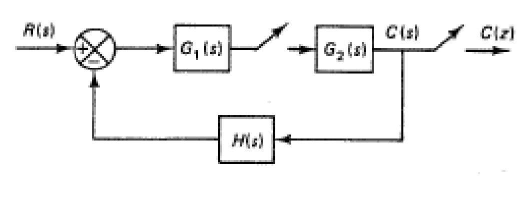
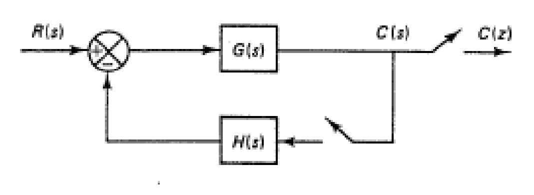
then taking the stared Laplace transform:

$$E^*(s) = R^*(s) - [GH(s)]^* E^*(s) \Rightarrow E^*(s) = \frac{R^*(s)}{1+GH^*(s)}$$

$$C^*(s) = G^*(s)E^*(s) = \frac{R^*(s)G^*(s)}{1+GH^*(s)}$$

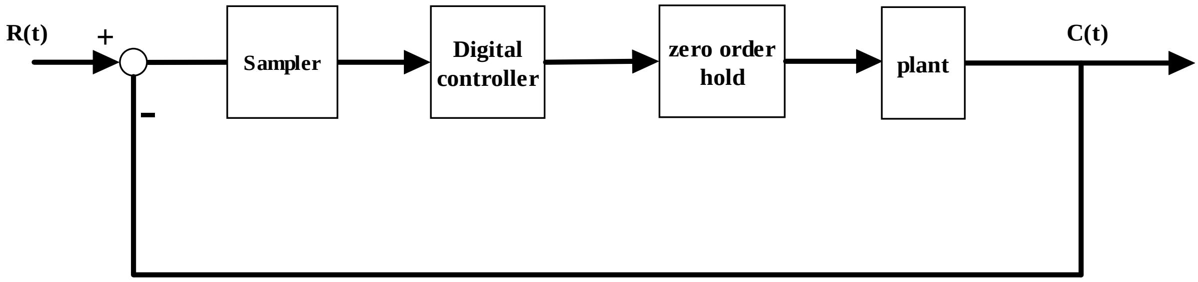
thus:

$$\Rightarrow C(z) = \frac{R(z)G(z)}{1+GH(z)} \Rightarrow \frac{C(z)}{R(z)} = \frac{G(z)}{1+GH(z)}$$

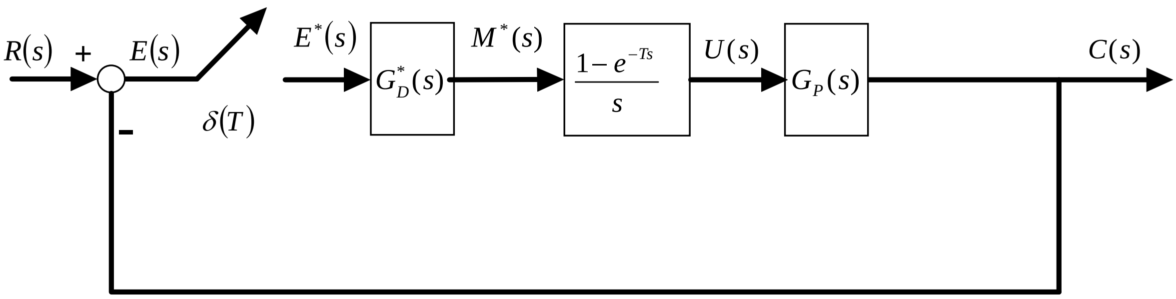
	$C(z) = \frac{G(z)R(z)}{1+GH(z)}$
	$C(z) = \frac{G(z)R(z)}{1+G(z)H(z)}$
	$C(z) = \frac{G_1(z)G_2(z)R(z)}{1+G_1(z)G_2H(z)}$
	$C(z) = \frac{G_2(z)G_1R(z)}{1+G_1G_2H(z)}$
	$C(z) = \frac{GR(z)}{1+GH(z)}$

Five configurations

Closed loop pulse transfer function of a digital control system



(a)



(b)

let's define $G(s) = \dfrac{1-e^{-Ts}}{s} G_P(s)$

then:

$$
\begin{aligned}
C(s) &= G(s)G_D^*(s)E^*(s) \Rightarrow \\
C(z) &= G(z)G_D(z)E(z) \\
&= G(z)G_D(z)\left(R(z) - C(z)\right) \Rightarrow \\
\frac{C(z)}{R(z)} &= \frac{G(z)G_D(z)}{1+G(z)G_D(z)}
\end{aligned}
\qquad 3.19
$$

Pulse transfer function of a digital PID controller

Analog PID controller is given by

$$m(t) = K\left[e(t) + \frac{1}{T_i}\int_o^t e(t)dt + T_d\frac{de(t)}{dt}\right], \qquad 3.20$$

where $e(t)$ is the input to the controller and $m(t)$ is the output of the controller.

Corresponding digital controller is:

$$M(z) = \left[K_P + \frac{K_I}{1-z^{-1}} + K_D\left(1-z^{-1}\right)\right]E(z) \qquad 3.21$$

where:

$$K_P = K - \frac{KT}{2T_i}$$

$$K_I = \frac{KT}{T_i}$$

$$K_D = \frac{KT_d}{T_i}$$

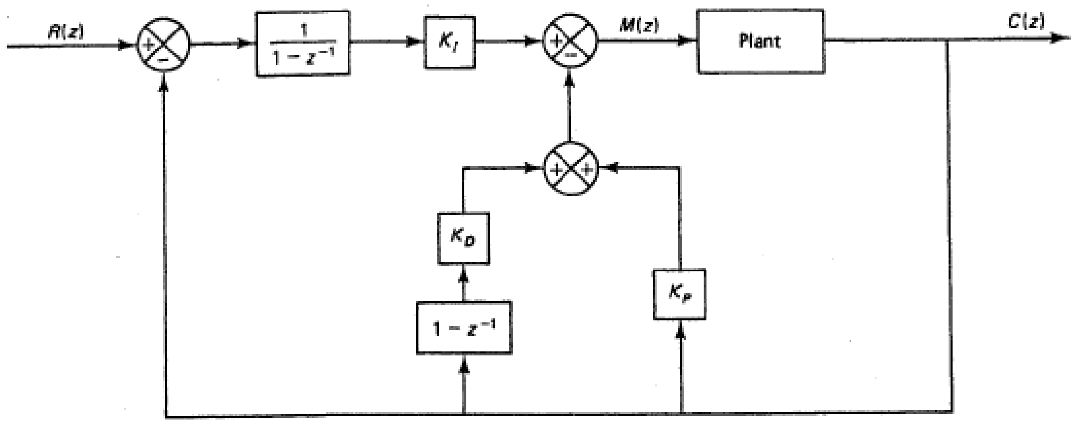
Thus $G_D(z) = \frac{M(z)}{E(z)} = \left[K_P + \frac{K_I}{1 - z^{-1}} + K_D\left(1 - z^{-1}\right)\right]$ 3.22

Note: 3.22 is referred to as the positional form of the PID control scheme.

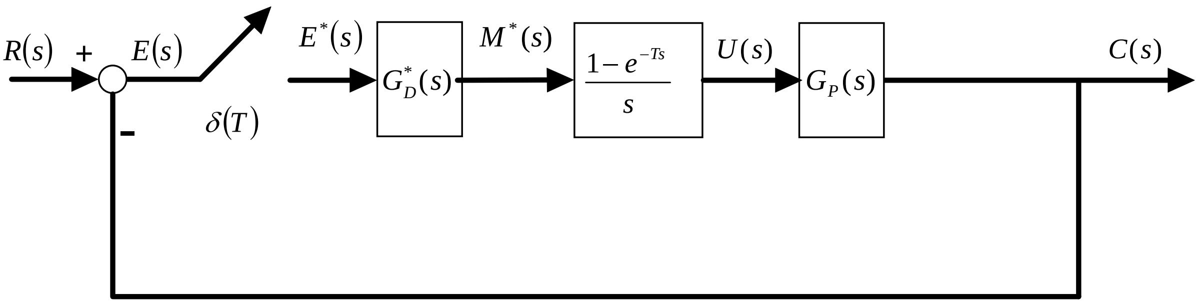
The velocity form of PID controller is given as follows:

$$M(z) = -K_P C(z) + K_I \frac{R(z) - C(z)}{1 - z^{-1}} + K_D\left(1 - z^{-1}\right)C(z) \qquad 3.23$$

Note: advantage of the velocity form is that initialization is not necessary when the operation is switched from manual to automatic.



Example 3.5 Design the digital controller for following system with $G_P(s) = \dfrac{1}{s+1}$



(b)

Assume the sampling time is 1 sec:

$$G(s) = \frac{1-e^{-Ts}}{s} G_P(s) = \frac{1-e^{-Ts}}{s} \frac{1}{s+1}$$

$$G(z) = Z\left[\frac{1-e^{-Ts}}{s}\frac{1}{s+1}\right] = \left(1-z^{-1}\right) Z\left(\frac{1}{s(s+1)}\right) = \frac{z^{-1}\left(1-e^{-T}\right)}{\left(1-e^{-T}z^{-1}\right)} = \frac{z^{-1}\left(1-e^{-1}\right)}{\left(1-e^{-1}z^{-1}\right)} = \frac{0.6321z^{-1}}{1-0.3679z^{-1}}$$

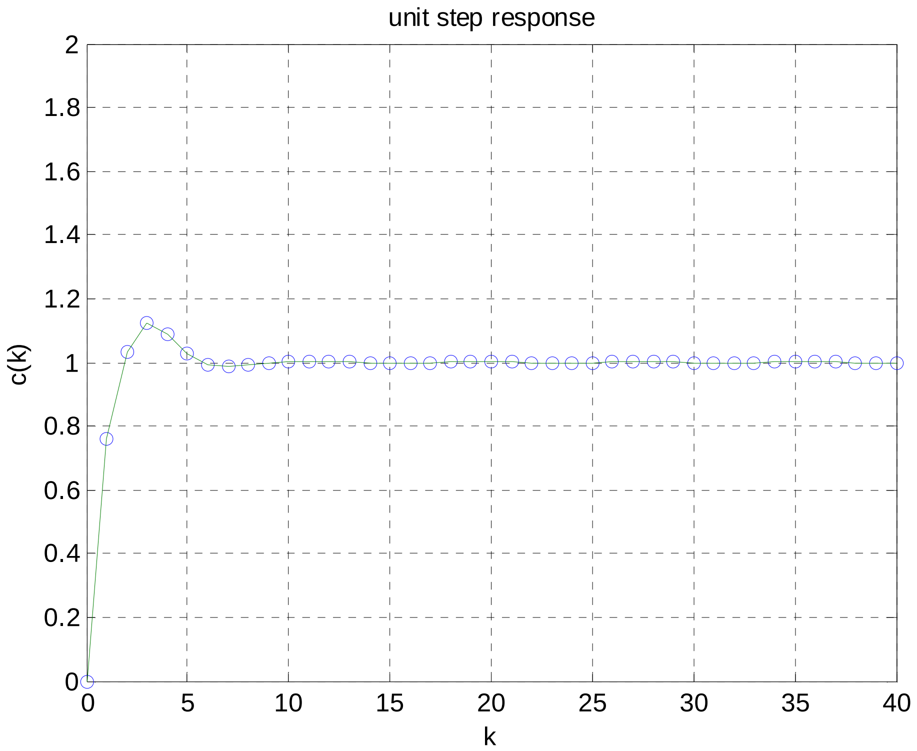
Design the digital controller with $K_P = 1, K_I = 0.1,\ K_D = 0.1$

$$G_D(z) = \frac{M(z)}{E(z)} = \left[K_P + \frac{K_I}{1-z^{-1}} + K_D\left(1-z^{-1}\right)\right] = 1 + \frac{0.1}{1-z^{-1}} + 0.1\left(1-z^{-1}\right) = \frac{1.2-0.3z^{-1}+0.1z^{-2}}{1-z^{-1}}$$

The closed loop pulse transfer function becomes:

$$\frac{C(z)}{R(z)} = \frac{G(z)G_D(z)}{1+G(z)G_D(z)} = \frac{\left(\dfrac{0.6321z^{-1}}{1-0.3679z^{-1}}\right)\left(\dfrac{1.2-0.3z^{-1}+0.1z^{-2}}{1-z^{-1}}\right)}{1+\left(\dfrac{0.6321z^{-1}}{1-0.3679z^{-1}}\right)\left(\dfrac{1.2-0.3z^{-1}+0.1z^{-2}}{1-z^{-1}}\right)}$$

$$= \frac{0.7585z^{-1} - 0.1896z^{-2} + 0.06321z^{-3}}{1-0.6094z^{-1} - 0.1783z^{-2} + 0.06321z^{-3}}$$

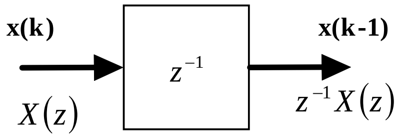


Remarks: PID parameter will be determined experimentally. Sampling period has to be chosen properly. Typically 10-30 sec for temperature control, 1-5sec for pressure control, 1-10 sec for liquid-level control.

III.6 Digital controller and filters

$$G(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_{m-1} z^{-(m-1)} + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots a_{n-1} z^{-(n-1)} + a_n z^{-n}}, \quad n \geq m \qquad 3.24$$

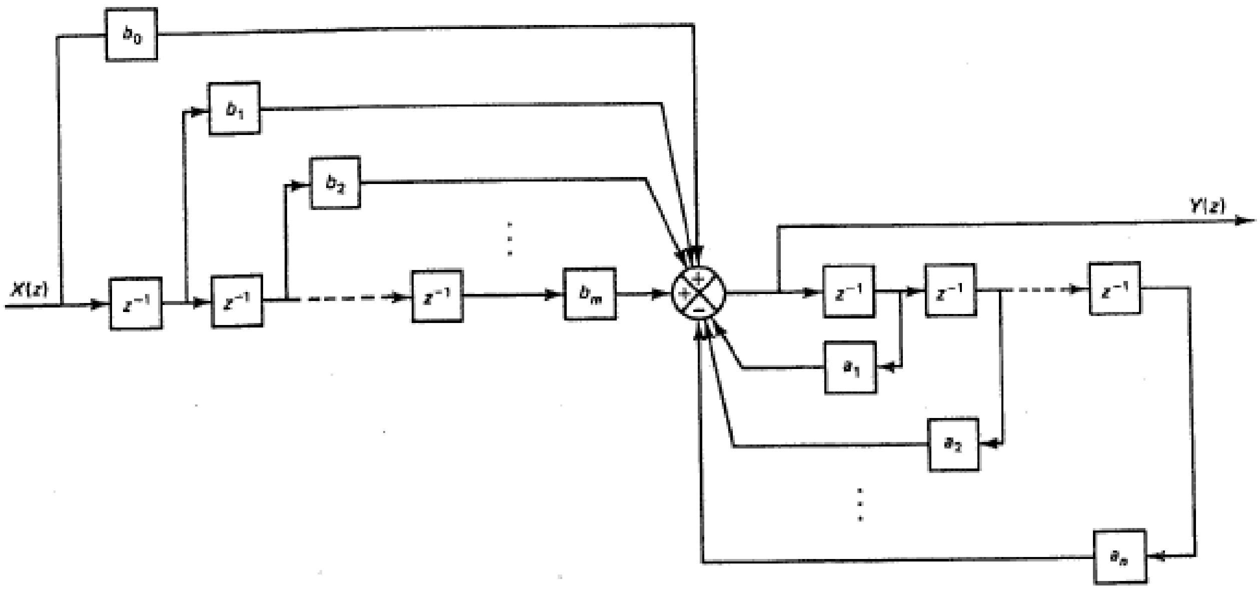
z^{-1} represents a delay of one time unit.



Direct Programming:

$$Y(z) = -a_1 z^{-1} Y(z) - \cdots a_{n-1} z^{-(n-1)} Y(z) - a_n z^{-n} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_{m-1} z^{-(m-1)} X(z) + b_m z^{-m} X(z)$$

Note: the total delay element is m+n



Standard Programming:

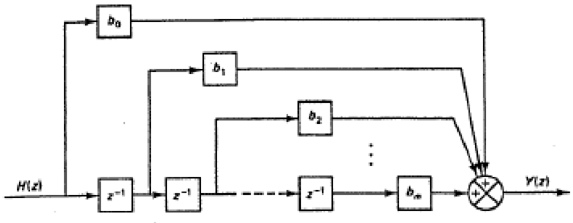
$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_{m-1} z^{-(m-1)} + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots a_{n-1} z^{-(n-1)} + a_n z^{-n}} = \frac{\left(b_0 + b_1 z^{-1} + \cdots + b_{m-1} z^{-(m-1)} + b_m z^{-m}\right) H(z)}{\left(1 + a_1 z^{-1} + \cdots a_{n-1} z^{-(n-1)} + a_n z^{-n}\right) H(z)}$$

$\Rightarrow$

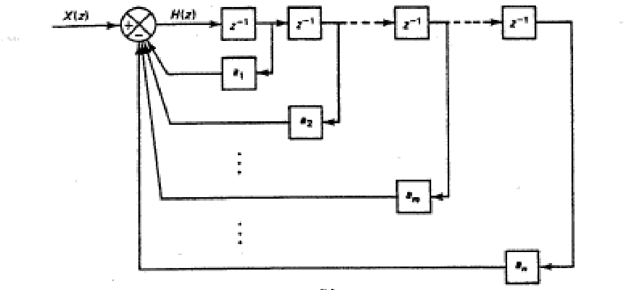
$$Y(z) = b_0 H(z) + b_1 z^{-1} H(z) + \cdots + b_{m-1} z^{-(m-1)} H(z) + b_m z^{-m} H(z)$$

$$X(z) = H(z) + a_1 z^{-1} H(z) + \cdots a_{n-1} z^{-(n-1)} H(z) + a_n z^{-n} H(z)$$

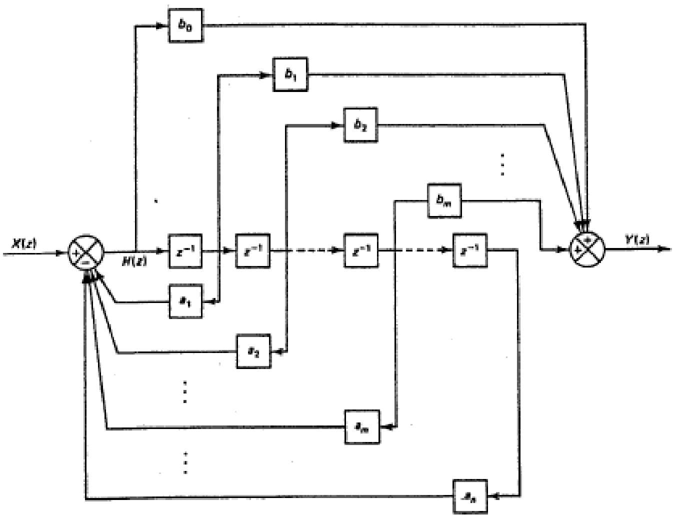
$$H(z) = X(z) - a_1 z^{-1} H(z) - \cdots a_{n-1} z^{-(n-1)} H(z) - a_n z^{-n} H(z)$$



(a)

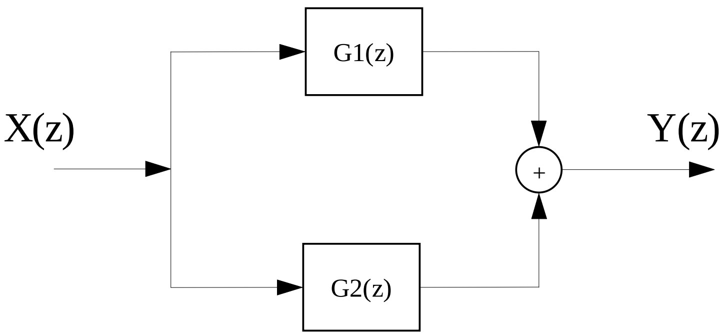


(b)

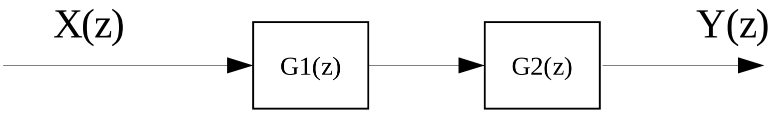


(c)

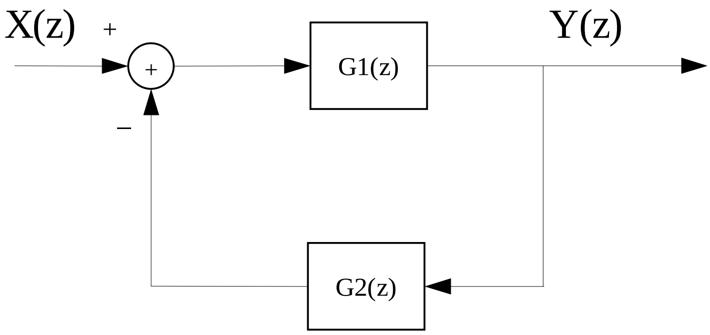
Interconnections



Parallel interconnection



Series connection



Feedback interconnection

Note: For digital controllers, it is important to minimize the delay elements, to minimize the summing points, to reduce the order of the pulse transfer function.

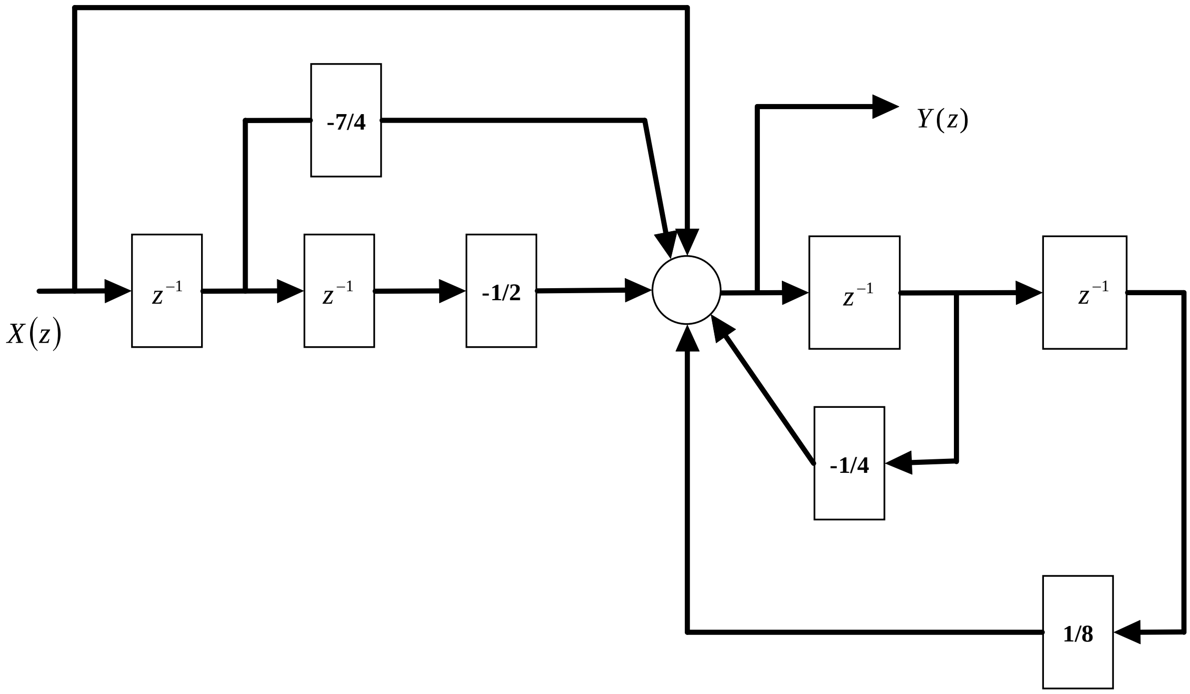
Example 3.6: Consider a causal LTI system function:

$$G(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}}$$

a) Draw a direct form block diagram for $G(z)$
b) Draw a cascade form block diagram for $G(z)$
c) Draw a parallel form block diagram for $G(z)$

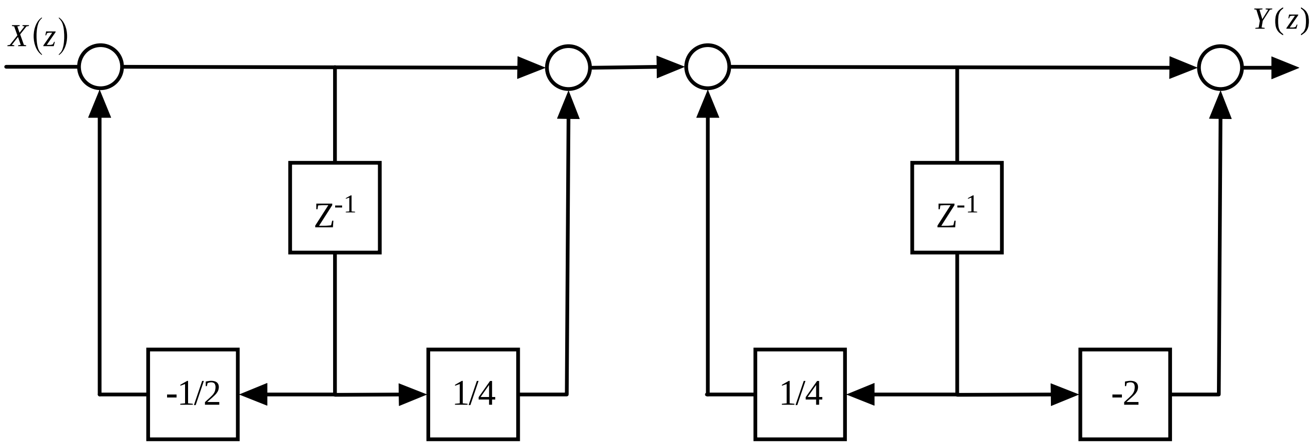
Solution:

a) $G(z)=\dfrac{Y(z)}{X(z)}=\dfrac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$



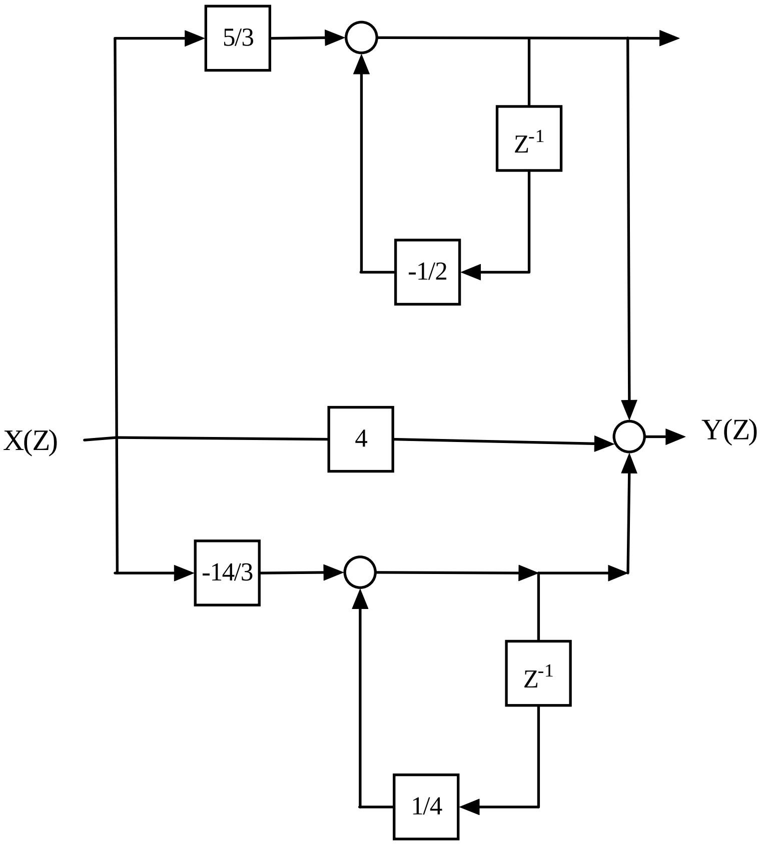
Direct form representation for the system

b) $G(z)=\dfrac{Y(z)}{X(z)}=\dfrac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}=\left(\dfrac{1+\frac{1}{4}z^{-1}}{1+\frac{1}{2}z^{-1}}\right)\left(\dfrac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}}\right)$



Cascade form block diagram

c) $G(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1 - \frac{7}{4}z^{-1} - \frac{1}{2}z^{-2}}{1 + \frac{1}{4}z^{-1} - \frac{1}{8}z^{-2}} = 4 + \dfrac{5/3}{1 + \frac{1}{2}z^{-1}} - \dfrac{14/3}{1 - \frac{1}{4}z^{-1}}$



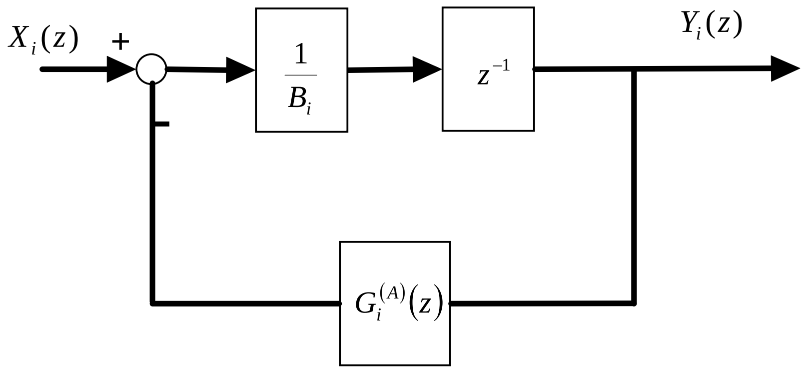
Parallel form block diagram

Ladder Programming: to avoid the coefficient sensitivity problem is to implement a ladder structure, to expand the $G(z)$ into following continued fraction form:

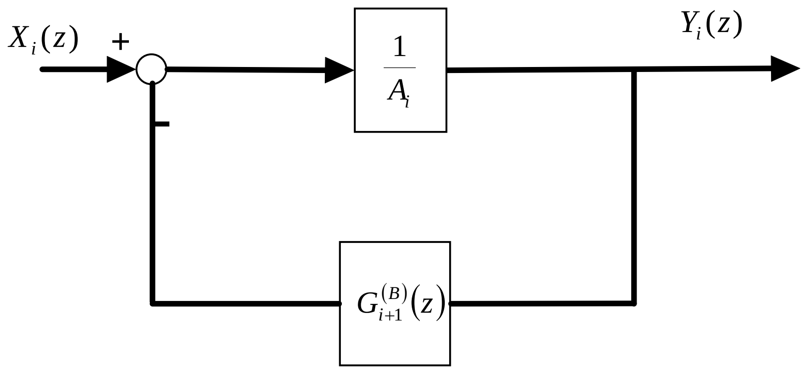
$$G(z) = A_0 + \cfrac{1}{B_1 z + \cfrac{1}{A_1 + \cfrac{1}{B_2 z + \cfrac{1}{\vdots \; A_{n-1} + \cfrac{1}{B_n z + \cfrac{1}{A_n}}}}}}$$

Let us define:

$$G_i^{(B)}(z) = \frac{1}{B_i z + G_i^{(A)}(z)} = \frac{\frac{1}{B_i} z^{-1}}{1 + \frac{1}{B_i} z^{-1} G_i^{(A)}(z)}, \quad i = 1,2,\cdots n-1$$

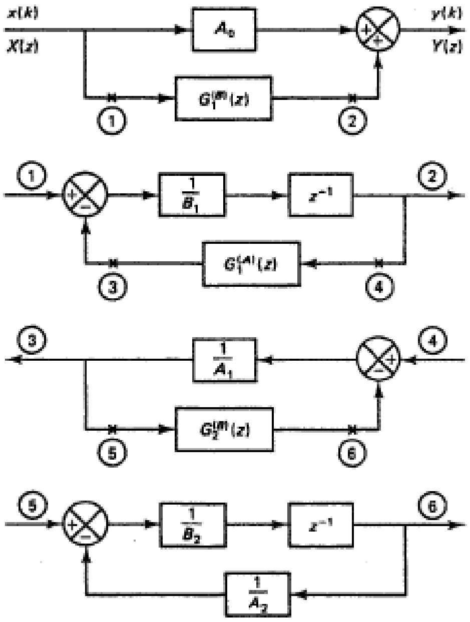


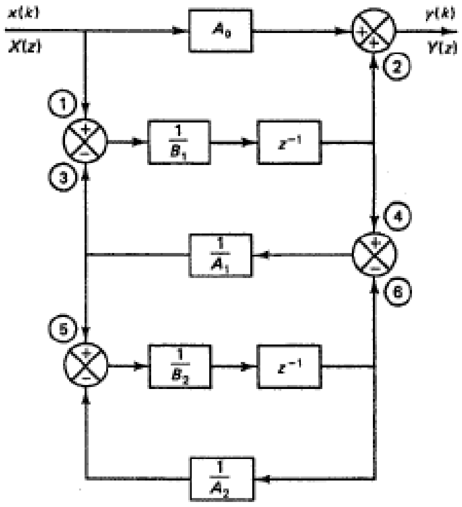
$$G_i^{(A)}(z) = \frac{1}{A_i + G_{i+1}^{(B)}(z)} = \frac{\frac{1}{A_i}}{1 + \frac{1}{A_i} G_{i+1}^{(B)}(z)}, \quad i = 1,2,\cdots n-1$$



$$G_n^{(B)}(z) = \frac{1}{B_n z + \frac{1}{A_n}}, \quad G_n^{(A)}(z) = \frac{1}{A_n}$$

Then $G(z) = A_0 + G_1^{(B)}(z)$





Note: Digital filters based on ladder programming have advantages with respect to coefficient sensitivity and accuracy. Ladder logic is not unique. It maybe be structures as a continued fraction expansion form around the origin in terms of z^{-1}

$$G(z) = \hat{A}_0 + \cfrac{1}{\hat{B}_1 z^{-1} + \cfrac{1}{\hat{A}_1 + \cfrac{1}{\hat{B}_2 z^{-1} + \cfrac{1}{\ddots \; \hat{A}_{n-1} + \cfrac{1}{\hat{B}_n z^{-1} + \cfrac{1}{\hat{A}_n}}}}}}$$

Example 3.7: Consider a causal LTI system function:

$$G(z)=\frac{Y(z)}{X(z)}=\frac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

Draw a block diagram for $G(z)$ using ladder programming.

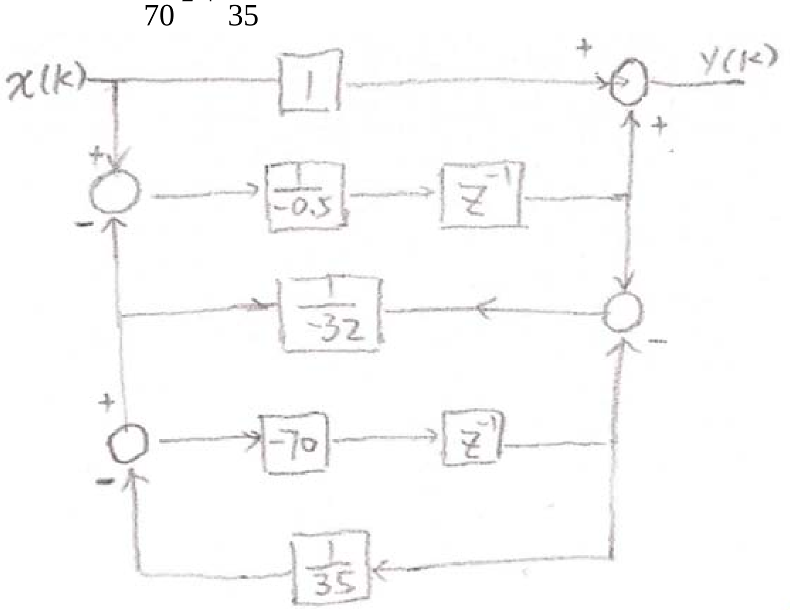
$$G(z)=\frac{Y(z)}{X(z)}=\frac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}=\frac{8z^2-14z-4}{8z^2+2z-1}$$

$$=1+\frac{-16z-3}{8z^2+2z-1}=1+\frac{1}{\frac{8z^2+2z-1}{-16z-3}}$$

$$=1+\frac{1}{-0.5z+\frac{0.5z-1}{-16z-3}}=1+\frac{1}{-0.5z+\frac{1}{\frac{-16z-3}{0.5z-1}}}$$

$$=1+\frac{1}{-0.5z+\frac{1}{-32+\frac{-35}{0.5z-1}}}=1+\frac{1}{-0.5z+\frac{1}{-32+\frac{1}{\frac{0.5z-1}{-35}}}}$$

$$=1+\frac{1}{-0.5z+\frac{1}{-32+\frac{1}{-\frac{1}{70}z+\frac{1}{35}}}}$$



Infinite impulse response filter and finite impulse response filter

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_{m-1} z^{-(m-1)} + b_m z^{-m}}{1 + a_1 z^{-1} + \cdots a_{n-1} z^{-(n-1)} + a_n z^{-n}}$$

$$Y(z) = -a_1 z^{-1} Y(z) - \cdots a_{n-1} z^{-(n-1)} Y(z) - a_n z^{-n} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_{m-1} z^{-(m-1)} X(z) + b_m z^{-m} X(z)$$

$$y(k) = -a_1 y(k-1) - \cdots a_{n-1} y(k-n+1) - a_n y(k-n) + b_0 x(k) + b_1 x(k-1) + \cdots + b_{m-1} x(k-m+1) + b_m x(k-m) \qquad 3.25$$

For equation 3.25, if not all a_i are zero, then the impulse response has an infinite number of nonzero samples. This filter is called **Infinite impulse response filter.**

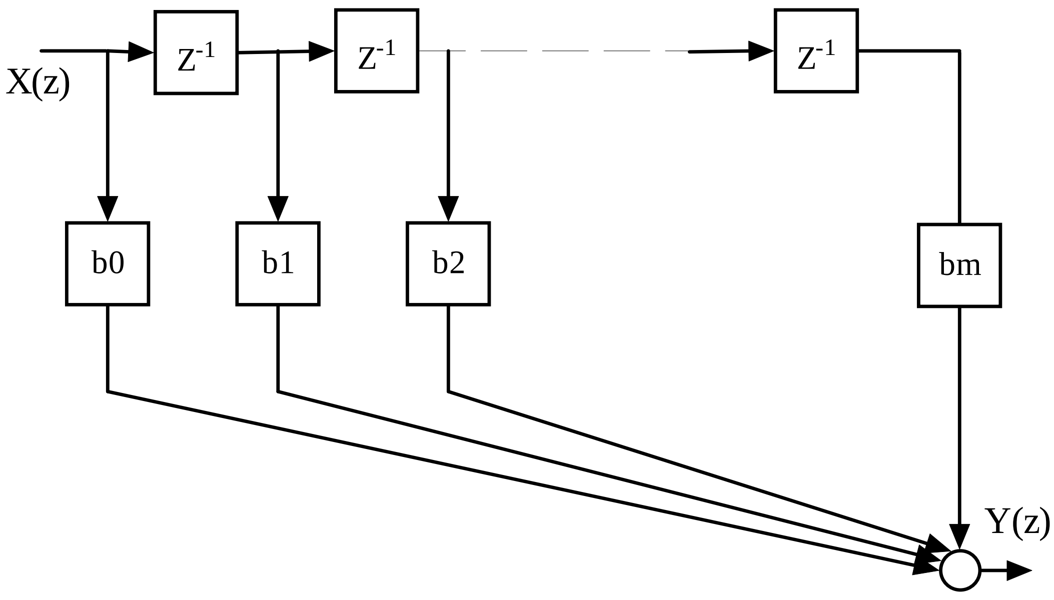
If all a_i are zero, then the impulse response has a finite number of nonzero samples. This filter is called **finite impulse response filter.** Which is following:

$$\frac{Y(z)}{X(z)} = b_0 + b_1 z^{-1} + \cdots + b_{m-1} z^{-(m-1)} + b_m z^{-m}$$

and

$$y(k) = b_0 x(k) + b_1 x(k-1) + \cdots + b_{m-1} x(k-m+1) + b_m x(k-m) \qquad 3.26$$

Realization of **finite impulse response filter:**



Note:

1) Finite impulse response filter (FIR) is nonrecursive. accumulation error can be avoided.
2) no feedback, direct programming and standard programming are same
3) Poles of pulse transfer function of FIR filter are at the origin. stable.
4) If the input signal involves high frequency components, then the delay elements needed in the finite impulse response filter increases. (disadvantage)