VII Polynomial Equations Approach to Control System Design

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VII.1. Introduction

Note: 1) Chapter six used state feedback approach
2) Polynomial equations approach is an alternative approach to the design via pole placement with a minimum order state observer. 3) This approach can be applied to MIMO system.

VII.2. Diophantine Equation

Consider the system:

$$\frac{Y(z)}{X(z)} = \frac{B(z)}{A(z)} \qquad 7.1$$

Where

$A(z) = z^n + a_1 z^{n-1} + \cdots a_{n-1} z + a_n$, $B(z) = b_0 z^n + b_1 z^{n-1} + \cdots b_{n-1} z + b_n$

Assume the system is completely controllable and observable.
There is no pole-zero cancellation in the pulse transfer function.
When $A(z)$ and $B(z)$ have no cancellation, these polynomials are called coprime polynomials.

A polynomial in z is called monic if the coefficient of the highest-degree term is unity. Thus $A(z)$ is monic.

Next define a $(2n-1)th$ degree polynomial $D(z)$ as follows:

$D(z) = d_0 z^{2n-1} + d_1 z^{2n-2} + \cdots d_{2n-2} z + d_{2n-1}$

Then there exist unique $(n-1)th$ degree polynomials $\alpha(z)$ and $\beta(z)$ such that

$$\alpha(z)A(z) + \beta(z)B(z) = D(z) \qquad 7.2$$

Where

$\alpha(z) = \alpha_0 z^{n-1} + \alpha_1 z^{n-2} + \cdots \alpha_{n-2} z + \alpha_{n-1}$, $\beta(z) = \beta_0 z^{n-1} + \beta_1 z^{n-2} + \cdots \beta_{n-2} z + \beta_{n-1}$

Eq. 7.2 is called Diophantine equation

Now let us define Sylvester matrix E, which is defined in terms of the coefficients of coprime polynomials $A(z)$ and $B(z)$.

$$E = \begin{bmatrix} a_n & 0 & \cdots & 0 & b_n & 0 & \cdots & 0 \\ a_{n-1} & a_n & \cdots & 0 & b_{n-1} & b_n & \cdots & 0 \\ \vdots & a_{n-1} & \cdots & 0 & \vdots & b_{n-1} & \cdots & 0 \\ a_1 & \vdots & & \vdots & b_1 & \vdots & & \vdots \\ 1 & a_1 & \cdots & a_{n-1} & b_0 & b_1 & \cdots & b_{n-1} \\ 0 & 1 & \cdots & a_{n-2} & 0 & b_0 & \cdots & b_{n-2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_1 & 0 & 0 & \cdots & b_1 \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & b_0 \end{bmatrix} \qquad 7.3$$

Note: 1) to use 7.3, $A(z)$ has to be monic.

2) Sylvester matrix E is nonsingular if and only if $A(z)$ and $B(z)$ are coprime.

Now define the vectors D and M such that:

$$D = \begin{bmatrix} d_{2n-1} \\ d_{2n-2} \\ \vdots \\ d_1 \\ d_0 \end{bmatrix}, M = \begin{bmatrix} \alpha_{n-1} \\ \alpha_{n-2} \\ \vdots \\ \alpha_0 \\ \beta_{n-1} \\ \beta_{n-2} \\ \vdots \\ \beta_0 \end{bmatrix}$$

Then the coefficients $\alpha_0, \alpha_1 \cdots \alpha_{n-1}$ and $\beta_0, \beta_1 \cdots \beta_{n-1}$ can be determined from $M = E^{-1}D$

Example 7.1 consider following $A(z)$ a monic polynomial of degree 3, $B(z)$ (a polynomial of degree 2) and $D(z)$ a polynomial of degree 5:

$$A(z) = z^3 + 2z^2 + 3z + 4$$
$$B(z) = 2z^2 + z + 5$$
$$D(z) = 5z^5 + 4z^4$$

$$E = \begin{bmatrix} 4 & 0 & 0 & 5 & 0 & 0 \\ 3 & 4 & 0 & 1 & 5 & 0 \\ 2 & 3 & 4 & 2 & 1 & 5 \\ 1 & 2 & 3 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4 \\ 5 \end{bmatrix}, \text{ thus } M = E^{-1}D = \begin{bmatrix} -77.5 \\ 27 \\ 5 \\ 62 \\ 12.5 \\ -16.5 \end{bmatrix}$$

$\alpha_0, \alpha_1, \alpha_2 = 5, 27, -77.5$, $\beta_0, \beta_1, \beta_2 = -16.5, 12.5, 62$,

Thus we have

$$A(z) = z^3 + 2z^2 + 3z + 4$$
$$B(z) = 2z^2 + z + 5$$
$$\alpha(z) = 5z^2 + 27z - 77.5$$
$$\beta(z) = -16.5z^2 + 12.5z + 62$$

It can be verified that $\alpha(z)A(z) + \beta(z)B(z) = D(z)$

Regular system design example 7.2.

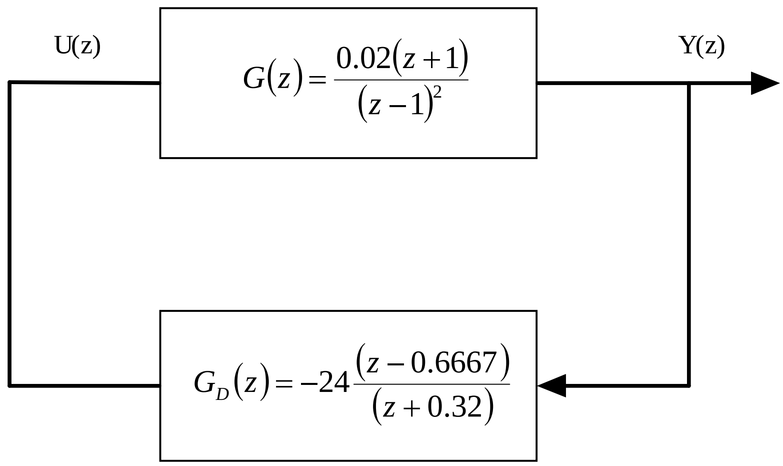


Figure 7.1

System transfer function is $G(z)=\frac{0.02(z+1)}{(z-1)^2}$, system is completely controllable and observable. The controller is designed to place the desired closed loop pole at

$z_1=0.6+j0.4$, $z_2=0.6-j0.4$

The controller is designed using pole placement technique as $G_D(z)=-24\frac{(z-0.6667)}{(z+0.32)}$

Next we will present the polynomial approach to have the same controller designed.

Consider the block diagram in figure 7.2 The feedback pulse transfer function $\frac{\beta(z)}{\alpha(z)}$

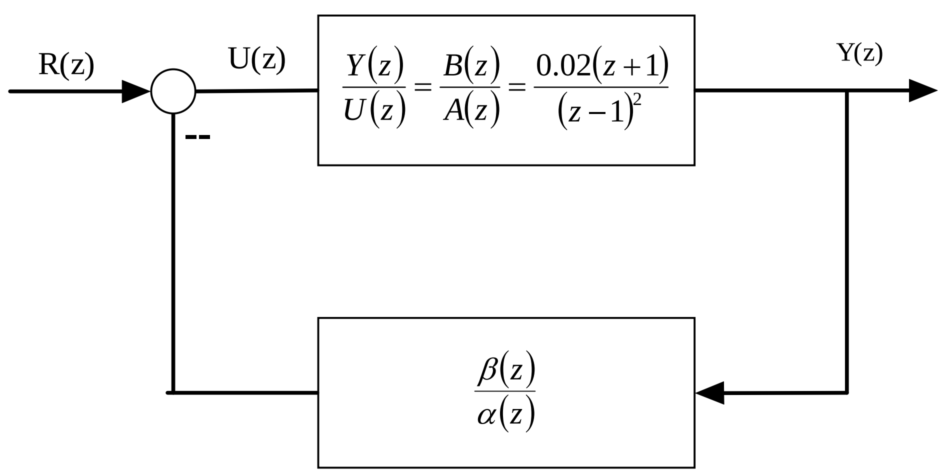


Figure 7.2

serves as a regulator. And the plant transfer function is $\frac{Y(z)}{U(z)}=\frac{B(z)}{A(z)}=\frac{0.02(z+1)}{(z-1)^2}$

Then the closed loop transfer function can be given by

$$\frac{Y(z)}{R(z)} = \frac{\frac{B(z)}{A(z)}}{1+\frac{B(z)}{A(z)}\frac{\beta(z)}{\alpha(z)}} = \frac{\alpha(z)B(z)}{\alpha(z)A(z)+\beta(z)B(z)} = \frac{\alpha(z)0.02(z+1)}{\alpha(z)(z-1)^2+\beta(z)0.02(z+1)}$$

Since the desired closed loop pole at $z_1 = 0.6 + j0.4$, $z_2 = 0.6 - j0.4$

$H(z) = (z-0.6-j0.4)(z-0.6+j0.4) = z^2 - 1.2z + 0.52$

The desired minimum order observer error polynomial was $F(z) = z$

To determine $\alpha(z), and\ \beta(z)$, we solve the Diophantine equation:

$$\alpha(z)A(z) + \beta(z)B(z) = F(z)H(z) = D(z) = z^3 - 1.2z^2 + 0.52z$$

Here $A(z) = (z-1)^2 = z^2 - 2z + 1$, and $B(z) = 0.02(z+1)$

We will have Sylvester matrix E

$$E = \begin{bmatrix} 1 & 0 & 0.02 & 0 \\ -2 & 1 & 0.02 & 0.02 \\ 1 & -2 & 0 & 0.02 \\ 0 & 1 & 0 & 0 \end{bmatrix}, D = \begin{bmatrix} 0 \\ 0.52 \\ -1.2 \\ 1 \end{bmatrix}, M = E^{-1}D = \begin{bmatrix} 0.32 \\ 1 \\ -16 \\ 24 \end{bmatrix}$$

Thus

$\alpha(z) = z + 0.32$, $\beta(z) = 24z - 16$, and the feedback is obtained as $\frac{\beta(z)}{\alpha(z)} = \frac{24(z-0.6667)}{z+0.32}$

VII.3. Polynomial Equations Approach to Control System Design

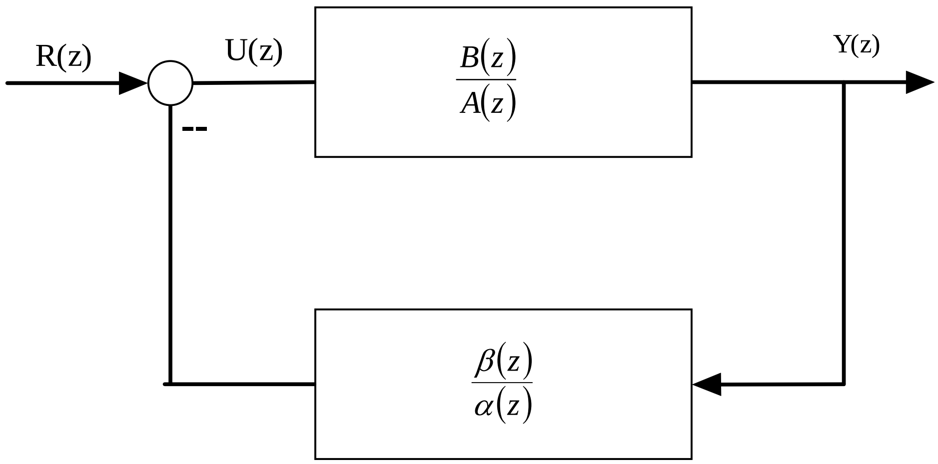


Figure 7.3. Block diagram of regulator system

The above controller is designed based on the Diophantine equation:

$\alpha(z)A(z)+\beta(z)B(z)=F(z)H(z)=D(z)$

Where $A(z)$ is monic polynomial of degree n, $B(z)$ is a polynomial of degree m ($m \le n$) $H(z)$ is the desired characteristic polynomial for the pole placement part and $F(z)$ is the characteristic polynomial for the minimum order observer. (both $H(z)$ and $F(z)$ are stable polynomials) the degree of $H(z)$polynomial is n and n-1 for $F(z)$.

Control system configuration 1

In figure 7.4, the output will follow the reference input. K0 is set that the steady state output $y(k)$ is equal to unity when the input $r(k)$ is a unit step sequence.

The closed loop transfer function is

$$\frac{Y(z)}{R(z)}=K_0\frac{\frac{B(z)}{A(z)}}{1+\frac{B(z)}{A(z)}\frac{\beta(z)}{\alpha(z)}}=K_0\frac{\alpha(z)B(z)}{\alpha(z)A(z)+\beta(z)B(z)}=K_0\frac{\alpha(z)B(z)}{H(z)F(z)}$$

To determine K_0, we set

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(1-z^{-1})y(z) = \lim_{z\to 1}(1-z^{-1})K_0 \frac{\alpha(z)B(z)}{H(z)F(z)}\frac{1}{1-z^{-1}} = K_0\frac{\alpha(1)B(1)}{H(1)F(1)} = 1$$

$$\Rightarrow K_0 = \frac{H(1)F(1)}{\alpha(1)B(1)}$$

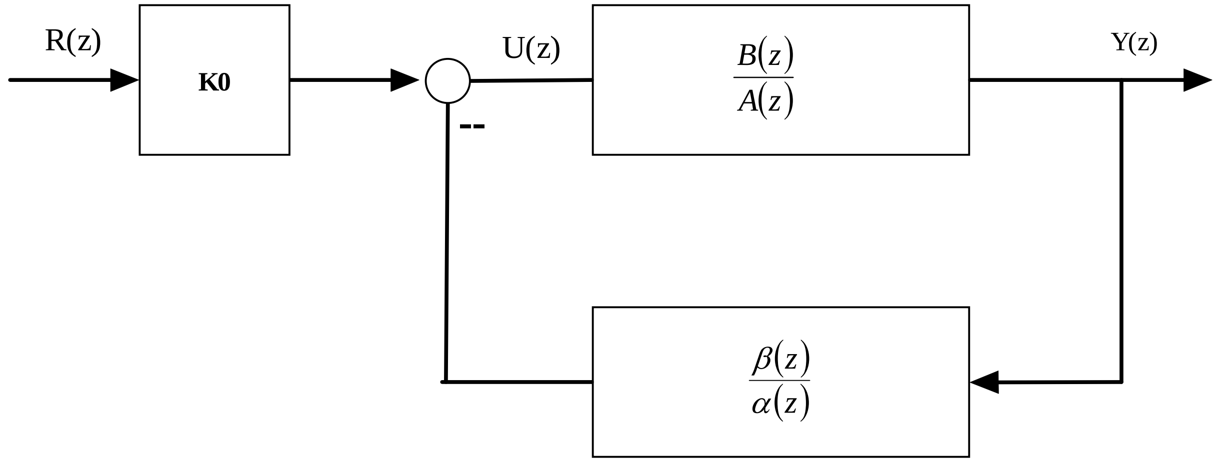


Figure 7.4

Control system configuration 2.

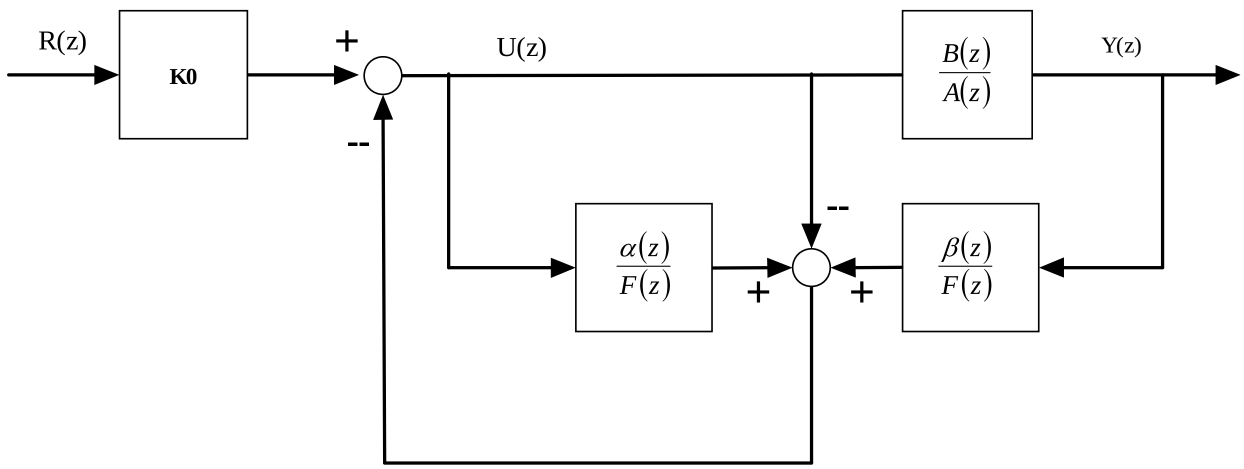


Figure 7.5

$$U(z) = -\left[\frac{\alpha(z)}{F(z)}U(z) - U(z) + \frac{\beta(z)}{F(z)}Y(z)\right] + K_0R(z)$$

$$\Rightarrow \frac{\alpha(z)}{F(z)}U(z) = -\frac{\beta(z)}{F(z)}Y(z) + K_0R(z) \qquad 7.4$$

The pulse transfer function is $\frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)} \Rightarrow U(z) = \frac{A(z)}{B(z)}Y(z)$ 7.5

By substituting eq. 7.5 into eq. 7.4, we obtain:

$$\frac{\alpha(z)}{F(z)}U(z) = \frac{\alpha(z)}{F(z)}\frac{A(z)}{B(z)}Y(z) = -\frac{\beta(z)}{F(z)}Y(z) + K_0 R(z)$$

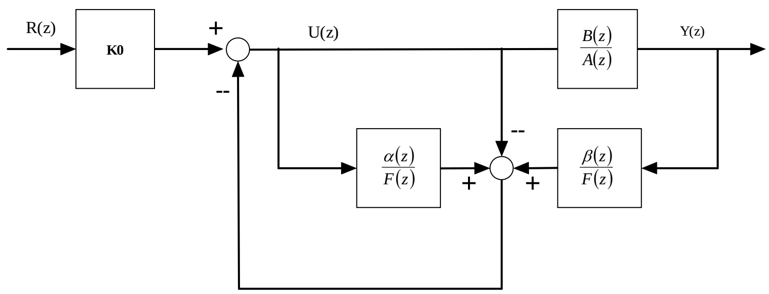
$$\Rightarrow \left(\frac{\alpha(z)}{F(z)}\frac{A(z)}{B(z)} + \frac{\beta(z)}{F(z)}\right)Y(z) = K_0 R(z) \qquad 7.6$$

$$\Rightarrow \frac{Y(z)}{R(z)} = \frac{K_0 F(z)B(z)}{\alpha(z)A(z) + \beta(z)B(z)} = \frac{K_0 F(z)B(z)}{H(z)F(z)} = \frac{K_0 B(z)}{H(z)}$$

Example 7.3 Consider the system with $\frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)} = \frac{1}{z^3 - 0.84z + 0.16}$

Where $A(z) = z^3 - 0.84z + 0.16$, and $B(z) = 1$

Use the block diagram configuration shown below



Assume the following $H(z) = z^3$, and $F(z) = z^2$.

Using the polynomial equations approach, design a control system for the plant. Obtain the unit step response and unit ramp response of the designed control system. Assume the sampling period T to be 1 sec.

Solution to the

$$\alpha(z)A(z) + \beta(z)B(z) = F(z)H(z) = D(z)$$

Where $H(z) = z^3$, and $F(z) = z^2$.

$$\alpha(z) = z^2 + 0.84$$

$$\beta(z) = -0.16z^2 + 0.7056z - 0.1344$$

$$\Rightarrow \frac{Y(z)}{R(z)} = \frac{K_0 B(z)}{H(z)} = \frac{K_0}{z^3}$$

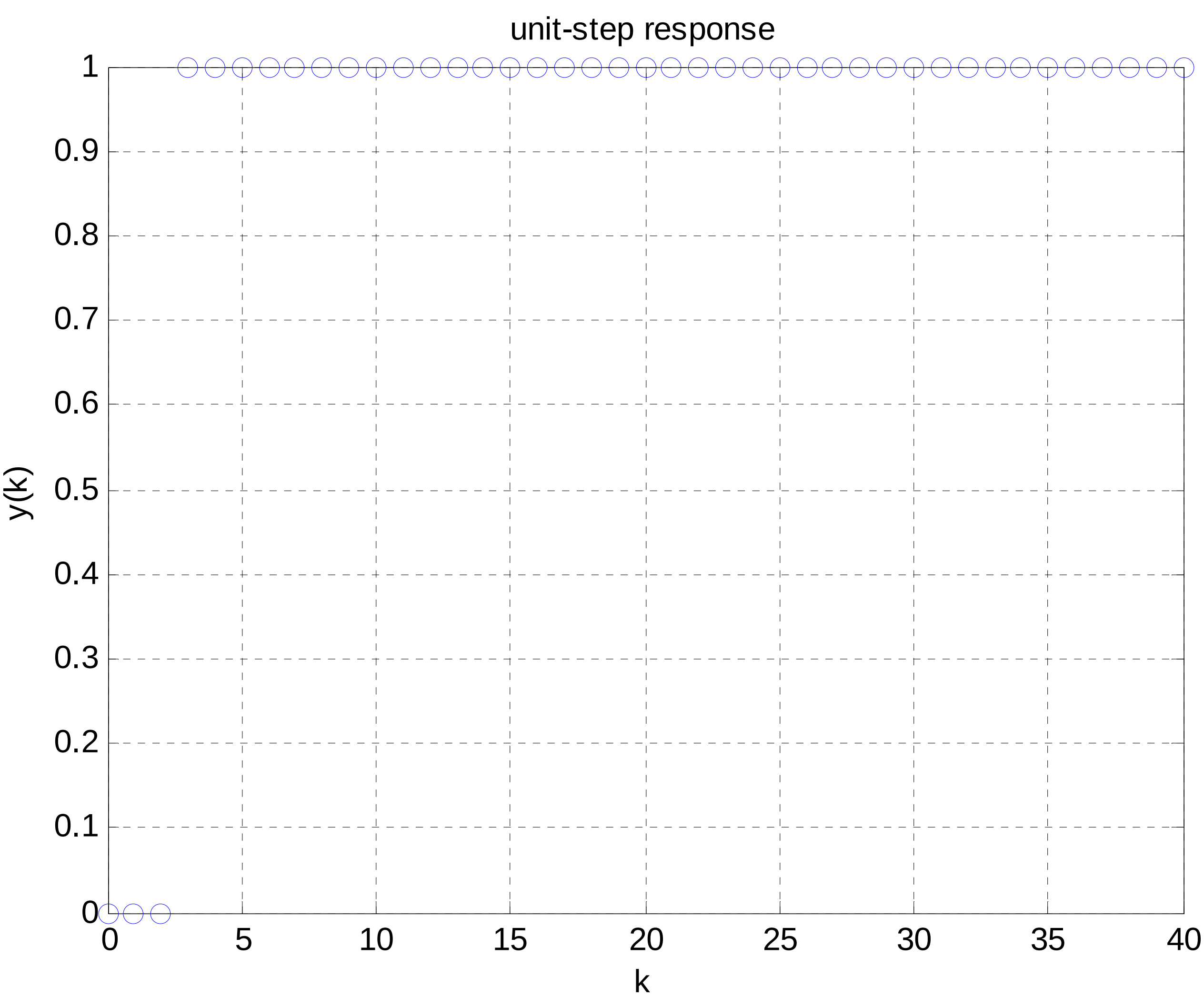
To determine K_0,we require $y(\infty)$ in the unit step response to be unity, or

$$\lim_{k\to\infty} y(k) = \lim_{z\to 1}(1 - z^{-1})y(z) = \lim_{z\to 1}(1 - z^{-1})\frac{K_0}{z^3}\frac{1}{1 - z^{-1}} = K_0 = 1$$

Thus $\frac{Y(z)}{R(z)} = \frac{1}{z^3}$, system has three closed loop poles at the origin. Thus it is a deadbeat system.

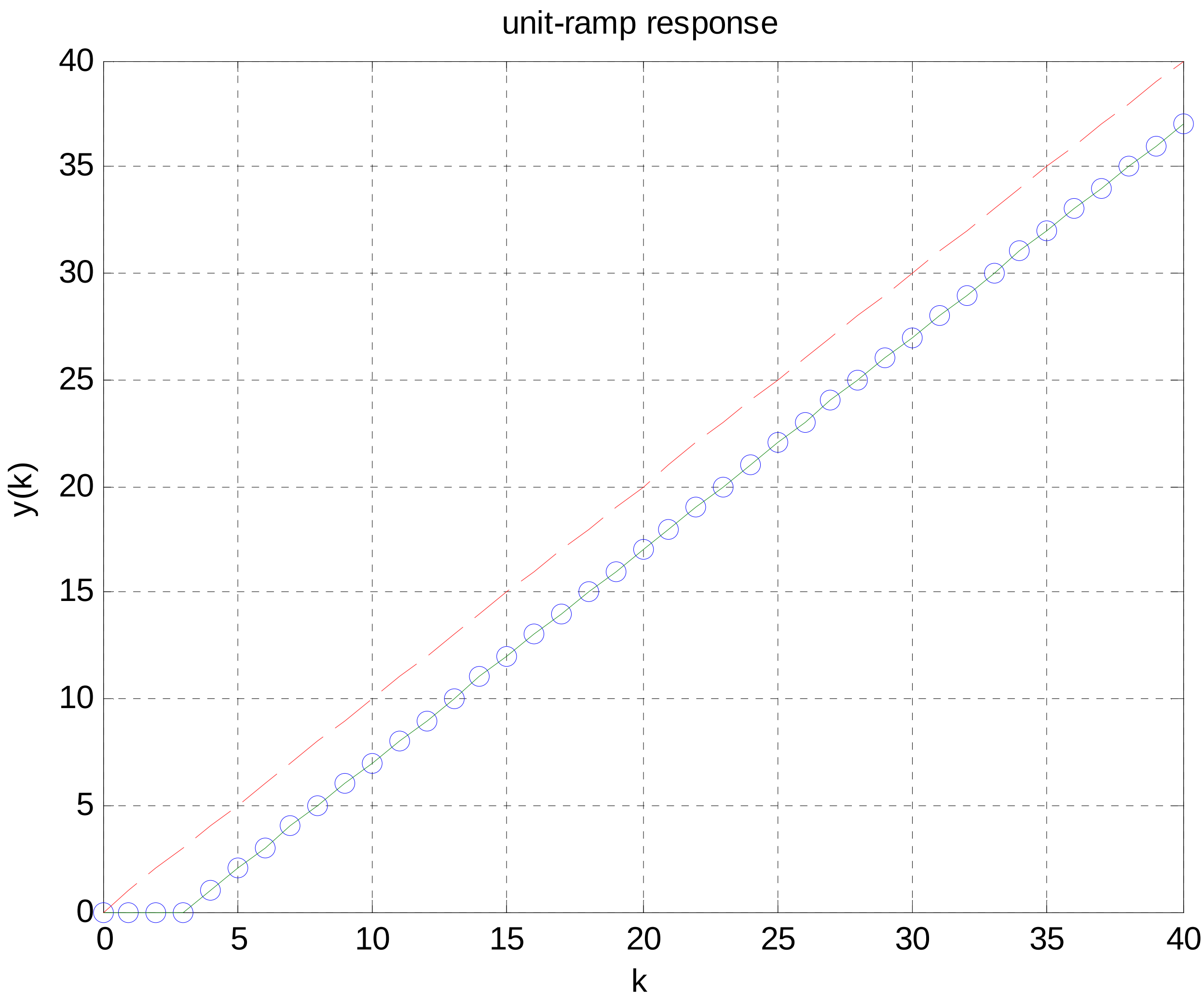
MATLAB code:

```
% unit step response
num= [ 0 0 0 1]
den=[1 0 0 0]
r=ones(1,41)
v=[0 40 0 1.6]
axis(v)
k=0:40
y=filter(num,den,r);
plot(k,y,'o');
grid
title('unit-step response')
xlabel('k')
ylabel('y(k)')
```



% unit ramp response

```
num= [ 0 0 0 1]
den=[1 0 0 0]
v=[0 40 0 40]
axis(v)
k=0:40
r=k
y=filter(num,den,r);
plot(k,y,'o',k,y,'-',k,k,'--');
grid
title('unit-ramp response')
xlabel('k')
ylabel('y(k)')
```



VII.4. Design of Model Matching Control System

Suppose the pulse transfer function of the plant is

$$\frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)}$$

Where $A(z)$ is monic polynomial of degree n, $B(z)$ is a polynomial of degree m ($m \le n$)
It may be possible to choose $H(z)$ such that it includes polynomial $B(z)$, or
$H(z) = B(z)H_1(z)$

Referring to equation 7.6

$$\frac{Y(z)}{R(z)} = \frac{K_0 B(z)}{H(z)} = \frac{K_0 B(z)}{B(z)H_1(z)} = \frac{K_0}{H_1(z)}$$

Thus, we eliminated the zeros of the numerator polynomial, which means that we can eliminate the zeros of the plant if we so desire.

Suppose we want the desired close loop system to be

$$\frac{Y(z)}{R(z)} = G_{\mathrm{model}} = \frac{B_m(z)}{A_m(z)},$$

It is possible to design such a system by use of the polynomial equations approach. Since we force the pulse transfer function of the control system exactly like the model, we call such control system a model matching control system.

Let's choose a stable polynomial of degree $n - m$ as $H_1(z)$. ($H_1(z)$ must be a stable polynomial)
Such that $H(z) = B(z)H_1(z)$

Refer to the Block diagram of model matching control system, assume that $\frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)}$ is completely controllable and observable.

We determine $\alpha(z)$, *and* $\beta(z)$ by solving following Diophantine equation

$$\alpha(z)A(z) + \beta(z)B(z) = F(z)B(z)H_1(z)$$

Where $F(z)$ is a stable polynomial of n-1 degree.

From the model matching control system diagram:

$$U(z) = -\left[\frac{\alpha(z)}{F(z)}U(z) - U(z) + \frac{\beta(z)}{F(z)}Y(z)\right] + V(z)$$
$$\Rightarrow \frac{\alpha(z)}{F(z)}U(z) + \frac{\beta(z)}{F(z)}Y(z) = V(z) \qquad 7.7$$

Since $\frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)} \Rightarrow U(z) = \frac{A(z)}{B(z)}Y(z)$ The pulse transfer function is

$$\frac{\alpha(z)}{F(z)}\frac{A(z)}{B(z)}Y(z) + \frac{\beta(z)}{F(z)}Y(z) = V(z)$$

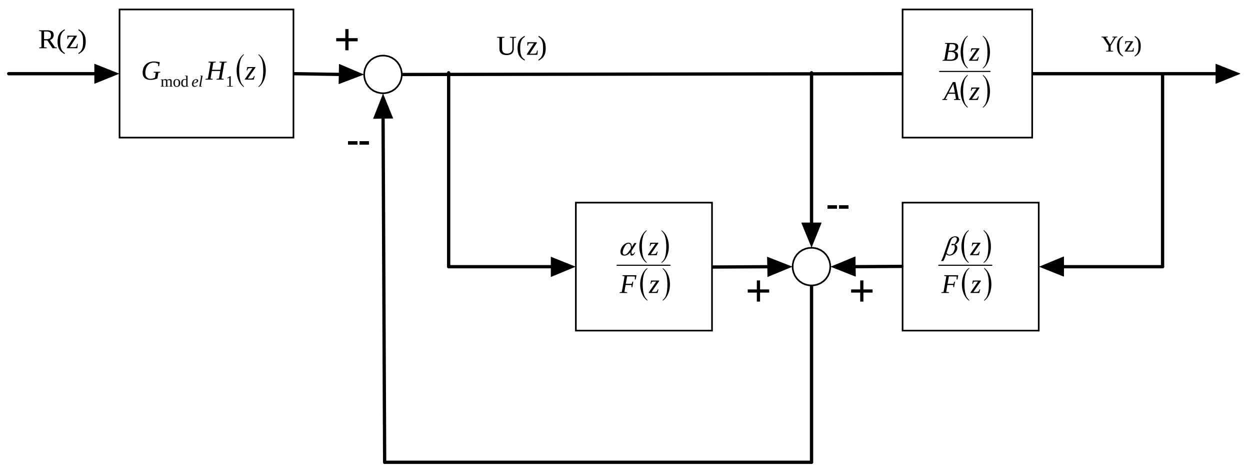
We have $\Rightarrow \frac{Y(z)}{V(z)} = \frac{F(z)B(z)}{\alpha(z)A(z) + \beta(z)B(z)} = \frac{F(z)B(z)}{F(z)B(z)H_1(z)} = \frac{1}{H_1(z)}$

Also $V(z) = G_{\text{mod}\,el}H_1(z)R(z)$

Hence: $\frac{Y(z)}{R(z)} = \frac{Y(z)}{V(z)}\frac{V(z)}{R(z)} = \frac{1}{H_1(z)}G_{\text{mod}\,el}H_1(z) = G_{\text{mod}\,el}$

Remarks: 1) $G_{\text{mod}\,el}H_1(z)$ has to be physically realizable, which means the order of the numerator will be less than the order of the denominator.

2) The numerator polynomial of the $B(z)$ must be stable because of the cancellation.



Block diagram of model matching control system

Example 7.4 consider the plant defined by

$$\frac{Y(z)}{U(z)} = \frac{B(z)}{A(z)} = \frac{0.01873(z+0.9356)}{(z-1)(z-0.8187)}$$

Using the polynomial equations approach, design a control system such that the system will behave like the following model, G_{model}

$$G_{model} = \frac{0.32}{z^2 - 1.2z + 0.52}$$

Obtain the unit step response and unit ramp response of the control system. The sampling period is 0.2 sec.

We shall assume that the designed system has the same block diagram as that of Figure 7-9. For the given plant,

$$\frac{Y(z)}{U(z)} = \frac{0.01873z + 0.01752}{z^2 - 1.8187z + 0.8187} = \frac{B(z)}{A(z)}$$

Thus,

$$A(z) = z^2 - 1.8187z + 0.8187 = z^2 + a_1 z + a_2$$

$$B(z) = 0.01873z + 0.01752 = b_0 z^2 + b_1 z + b_2$$

Hence,

$$a_1 = -1.8187, \quad a_2 = 0.8187$$

$$b_0 = 0, \quad b_1 = 0.01873, \quad b_2 = 0.01752$$

[Clearly, there are no common factors between A(z) and B(z) and the numerator B(z) is a stable polynomial.]

In the design process we choose H(z) as the desired characteristic polynomial of degree 2. Let us choose a stable polynomial of degree 1 as $H_1(z)$, or

$$H_1(z) = z + 0.5$$

[Choice of $H_1(z)$ is, in a sense, arbitrary as long as it is a stable polynomial.] Now define

$$H(z) = B(z)H_1(z) = (0.01873z + 0.01752)(z + 0.5)$$

(This is the desired characteristic polynomial for this system.) Next, we choose

$$F(z) = z$$

[F(z) can be any stable first-degree polynomial.] Define

$$D(z) = F(z)B(z)H_1(z)$$
$$= z(0.01873z + 0.01752)(z + 0.5)$$
$$= 0.01873z^3 + 0.026885z^2 + 0.00876z$$

Hence,

$$d_0 = 0.01873, \quad d_1 = 0.026885, \quad d_2 = 0.00876, \quad d_3 = 0$$

Define

$$\alpha(z) = \alpha_0 z + \alpha_1$$
$$\beta(z) = \beta_0 z + \beta_1$$

We determine $\alpha(z)$ and $\beta(z)$ by solving the following Diophantine equation:

$$\alpha(z)A(z) + \beta(z)B(z) = F(z)B(z)H_1(z) = D(z)$$

or

$$\alpha(z)(z^2 - 1.8187z + 0.8187) + \beta(z)(0.01873z + 0.01752)$$
$$= 0.01873z^3 + 0.026885z^2 + 0.00876z$$

The 4 x 4 Sylvester matrix $\underset{\sim}{E}$ for this problem becomes as follows:

$$\underset{\sim}{E} = \begin{bmatrix} 0.8187 & 0 & 0.01752 & 0 \\ -1.8187 & 0.8187 & 0.01873 & 0.01752 \\ 1 & -1.8187 & 0 & 0.01873 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Define

$$\underset{\sim}{D} = \begin{bmatrix} d_3 \\ d_2 \\ d_1 \\ d_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.00876 \\ 0.026885 \\ 0.01873 \end{bmatrix}, \qquad \underset{\sim}{M} = \begin{bmatrix} \alpha_1 \\ \alpha_0 \\ \beta_1 \\ \beta_0 \end{bmatrix}$$

Then matrix $\underset{\sim}{M}$ can be determined from

$$\underset{\sim}{M} = \underset{\sim}{E}^{-1}\underset{\sim}{D}$$

A MATLAB solution for the determination of $\underset{\sim}{M}$ is given below.

```
E =

    0.8187         0    0.0175         0
   -1.8187    0.8187    0.0187    0.0175
    1.0000   -1.8187         0    0.0187
         0    1.0000         0         0

/ D = [0;0.00876;0.026885;0.01873];
/ format long
/ M = inv(E)*D

M =

   0.01752000000000
   0.01873000000000
  -0.81870000000000
   2.31870000000000
```

Hence

$$\alpha(z) = \alpha_0 z + \alpha_1 = 0.01873z + 0.01752$$

$$\beta(z) = \beta_0 z + \beta_1 = 2.3187z - 0.8187$$

Using $\alpha(z)$ and $\beta(z)$ thus determined, $Y(z)/V(z)$ becomes as follows:

$$\frac{Y(z)}{V(z)} = \frac{F(z)B(z)}{F(z)B(z)H_1(z)} = \frac{1}{H_1(z)} = \frac{1}{z + 0.5}$$

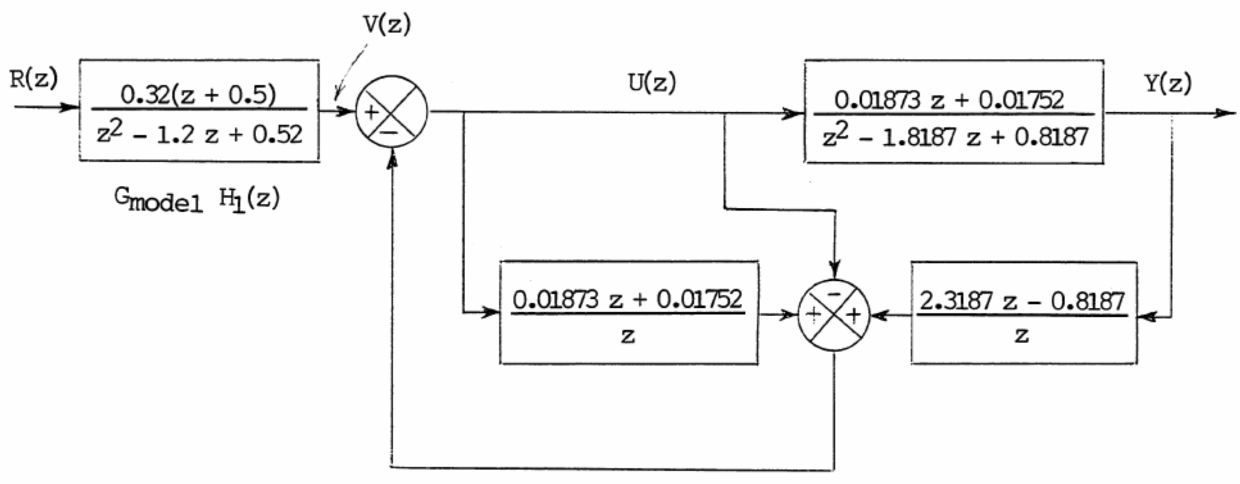
Since $V(z)/R(z)$ is

$$\frac{V(z)}{R(z)} = G_{model}\, H_1(z) = \frac{0.32}{z^2 - 1.2z + 0.52}\,(z + 0.5)$$

the pulse transfer function $Y(z)/R(z)$ becomes

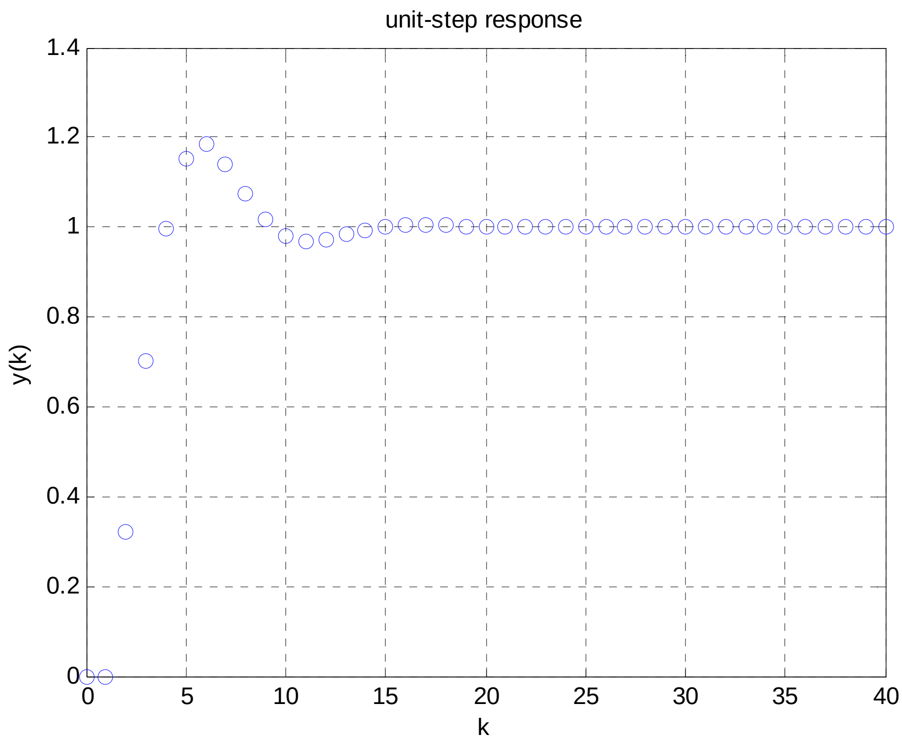
$$\frac{Y(z)}{R(z)} = \frac{Y(z)}{V(z)}\,\frac{V(z)}{R(z)} = \frac{0.32}{z^2 - 1.2z + 0.52} = G_{model}$$

The designed model-matching control system has the block diagram as shown in the next page.



Unit step response :

```
% step response
num= [ 0 0 0.32]
den=[1 -1.2 0.52]
r=ones(1,41)
v=[0 40 0 1.5]
axis(v)
k=0:40
y=filter(num,den,r);
plot(k,y,'o');
grid
title('unit-step response')
xlabel('k')
ylabel('y(k)')
```



```
% unit ramp response
num= [ 0 0 0.32]
den=[1 -1.2 0.52]

v=[0 20 0 4]
axis(v)
k=0:20
r=[0.2*k]
y=filter(num,den,r);
plot(k,y,'o',k,y,'-',k,0.2*k,'--');
grid
title('unit-ramp response')
xlabel('k')
ylabel('y(k)')
```